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On Passing the Buck

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Introduction

Imagine there are $n > 1$ people seated around a table, and person S starts with a fair coin they will flip to decide whom to hand the coin next – if "heads" they pass right, and if "tails" they pass left. This process continues until all people at the table have "touched" the coin. Curiously, it turns out that all people seated at the table other than S have the same probability $\frac{1}{n-1}$ of being last to touch the coin. In fact, Lovász and Winkler ("A note on the last new vertex visited by a random walk," J. Graph Theory, Vol. 17 Iss. 5 (1993), 593–596) showed that this situation and the one where a person is permitted to pass the coin to anyone else with uniform probability $\frac{1}{n-1}$ are the only scenarios where everyone at the table other than S have the same probability $\frac{1}{n-1}$ of touching the coin last. This begs the question – what is the probability that a person will touch the coin last in scenarios that lie outside these two? We consider a version where the table has two sides, and the "passing rule" involves handing the coin to someone on the opposite side of the table with uniform probability. What is the resulting probability that a particular person touches the coin last in this two-sided situation?

Definition 1

- Let $K_{2,n}$ denote the event where two people are on the left-hand side, and n people are on the right-hand side. The possible paths of the coin can be represented by Figure 1:

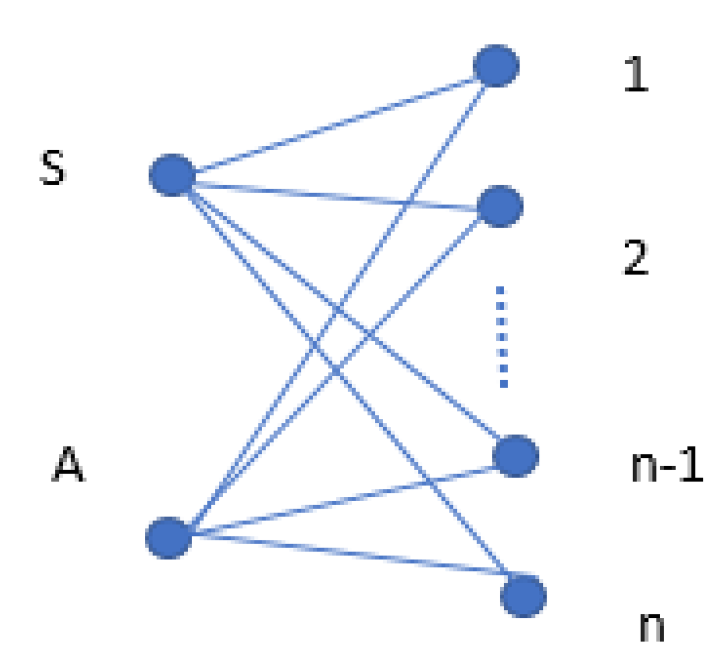


Figure 1: $K_{2,n}$ Graph

Definition 2

- In Figure 2, we denote the path of the coin as it is passed between people. For any ordered pair (m, n), m represents the number of people on the left-hand side who have touched the coin, and n represents the number of people who have touched the coin on the right hand side. The side that is underlined shows which side the coin is on. $(2, n)$ means that person A was last to touch the coin. The fraction on the top of the arrow indicates the probability of passing the coin in the indicated way.

$$(\underline{m}, n) \xrightarrow{f/p} (i, \underline{j})$$

Figure 2: Passing Probability Diagram

Method

In order to calculate the probability that A is last to receive the coin, $P(A \text{ last})$, we first calculate the probability that it reaches A with no repeats. The probability of this occurring is:

$$1 \times \tfrac{1}{2} \times \tfrac{n-1}{n} \times \tfrac{1}{2} \times \tfrac{n-2}{n} \times \tfrac{1}{2} \times \cdots \times \tfrac{3}{n} \times \tfrac{1}{2} \times \tfrac{2}{n} \times \tfrac{1}{2} \times \tfrac{1}{n} \times \tfrac{1}{2}$$
$$= \frac{(n-1)!}{2^n n^{n-1}}$$

Method Cont.

Now, we calculate the probabilities when paths are repeated using sums of infinite geometric series.

$$((\tfrac{1}{2} \times \tfrac{1}{n})^0 + (\tfrac{1}{2} \times \tfrac{1}{n})^1 + \cdots) \longleftarrow \frac{2n}{2n-1}$$
$$((\tfrac{1}{2} \times \tfrac{2}{n})^0 + (\tfrac{1}{2} \times \tfrac{2}{n})^1 + \cdots) \longleftarrow \frac{2n}{2n-2}$$
$$((\tfrac{1}{2} \times \tfrac{3}{n})^0 + (\tfrac{1}{2} \times \tfrac{3}{n})^1 + \cdots) \longleftarrow \frac{2n}{2n-3}$$
$$\cdots\cdots$$
$$((\tfrac{1}{2} \times \tfrac{n-1}{n})^0 + (\tfrac{1}{2} \times \tfrac{n-1}{n})^1 + \cdots) \longleftarrow \frac{2n}{n+1}$$
$$((\tfrac{1}{2} \times \tfrac{n}{n})^0 + (\tfrac{1}{2} \times \tfrac{n}{n})^1 + \cdots) \longleftarrow \frac{2n}{n}$$

The product of the probabilities of the repeated paths is:

$$\frac{2n}{(2n-1)} \times \frac{2n}{(2n-2)} \times \frac{2n}{2n-3} \times \cdots \times \frac{2n}{(n+1)} \times \frac{2n}{n}$$
$$= \frac{(2n)^n}{\frac{(2n-1)!}{(n-1)!}}$$

Multiplying $\frac{(n-1)!}{2^n n^{n-1}}$ by $\frac{(2n)^n}{\frac{(2n-1)!}{(n-1)!}}$ gives us our result:

$$\frac{(n-1)!}{2^n n^{n-1}} \times \frac{(2n)^n}{\frac{(2n-1)!}{(n-1)!}} = \frac{(n-1)! \times (n-1)! \times 2^n \times n^n}{2^n \times n^{n-1} \times (2n-1)!}$$
$$= \frac{(n-1)! \times (n-1)! \times \cancel{2^n} \times \cancel{n^n}}{\cancel{2^n} \times \cancel{n^{n-1}} \times (2n-1)!} = \frac{n!(n-1)!}{(2n-1)!}$$
$$= \frac{1}{\binom{2n-1}{n}}$$

$K_{2,n}$ Diagram

$$\begin{array}{cccccccccc}
(\underline{1},0) & \xrightarrow{n/n} & (1,\underline{1}) & \xrightarrow{1/2} & (\underline{2},1) & & & & & \\
 & & {\scriptstyle 1/2}\downarrow\uparrow{\scriptstyle 1/n} & & & & & & & \\
 & & (\underline{1},1) & \xrightarrow{(n-1)/n} & (1,\underline{2}) & \xrightarrow{1/2} & (\underline{2},2) & & & \\
 & & & & {\scriptstyle 1/2}\downarrow\uparrow{\scriptstyle 2/n} & & & & & \\
 & & & & (\underline{1},2) & \xrightarrow{(n-2)/n} & (1,\underline{3}) & \xrightarrow{1/2} & (\underline{2},3) & \\
 & & & & & & {\scriptstyle 1/2}\downarrow\uparrow{\scriptstyle 3/n} & & & \\
 & & & & & & (\underline{1},3) & \xrightarrow{(n-3)/(n)} & (1,\underline{4}) & \\
 & & & & & & & \cdots\cdots & & \\
 & & & & & & & & {\scriptstyle 1/2}\downarrow\uparrow{\scriptstyle (n-1)/n} & \\
 & & & & & & & & (\underline{1},n-1) \xrightarrow{1/n} (1,\underline{n}) & \xrightarrow{1/2} (\underline{2},n) \\
 & & & & & & & & {\scriptstyle 1/2}\downarrow\uparrow{\scriptstyle n/n} & \\
 & & & & & & & & (\underline{1},n) &
\end{array}$$

Figure 3: $K_{2,n}$ Probability Graph

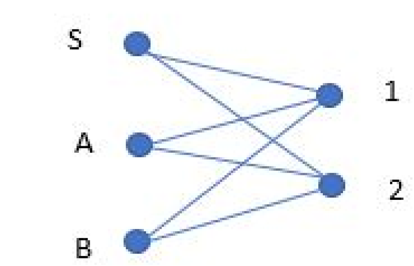
Results

For a $K_{2,n}$ graph, the formula for the probability of visiting A last is:

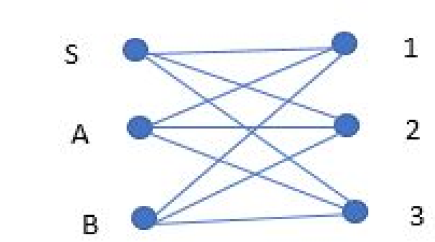
$$P_{2,n} = \frac{n!(n-1)!}{(2n-1)!} = \frac{1}{\binom{2n-1}{n}}$$

Future Work

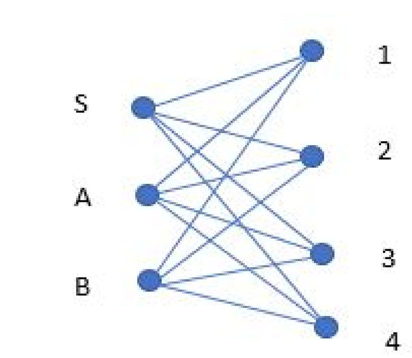
The probabilities of some $K_{3,n}$ graphs have been calculated by using the same passing diagram method.



$$P_{3,2} = \frac{4}{5}$$



$$P_{3,3} = \frac{59}{140}$$



$$P_{3,4} = \frac{32}{165}$$

Our next step is to find the pattern for $K_{3,n}$, $K_{4,n}$, and so on to hopefully find a generalized formula for $K_{m,n}$.

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