MATCH-MAKING FOR STABILITY

A Survey of the Stable Marriage Problem

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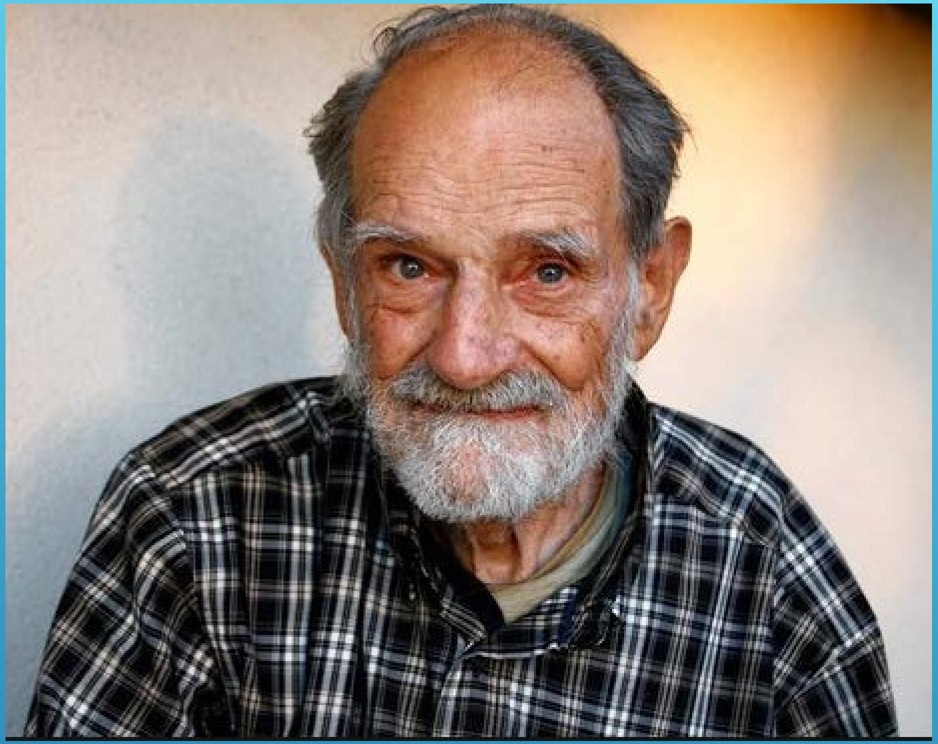

DAVID GALE & LLOYD SHAPLEY

COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

THE AMERICAN MATHEMATICAL MONTHLY, VOL. 69, NO. 1 (1962)

GALE & SHAPLEY

College Admissions and the Stability of Marriage

- G & S considered the problem of matching several students to several colleges, according to preferences of each, where the colleges each had a specific quota.
- An instance of "instability": there are two students α and β who are matched to colleges A and B, resp., but β prefers A to B and A prefers β to α
- A matching of students to colleges is considered "unstable" if there is any instance of instability. It is called "stable" otherwise.
- Can we always find a matching that is stable?

GALE & SHAPLEY

College Admissions and the Stability of Marriage

- Remarkably, stability is *always* achievable, no matter the preferences of students/colleges!
- G & S showed this by actually describing an algorithm (GS) that found a specific stable matching.
- To simplify the analysis, they initially changed the situation: *n* students and *n* colleges, each with a quota of 1.
- Like "marriages"!
- "students" = "men", "colleges" = "women"
- We will return to the original question later, but this is more fun.
- Here, an "unstable matching" means that there is some man a and some woman A that prefer each other to their assigned match.

Example 1. The following is the "ranking matrix" of three men, α, β, and γ, and three women, A, B, and C.

$$\begin{array}{c|ccc|} & A & B & C \\ \hline \alpha & 1,3 & 2,2 & 3,1 \\ \beta & 3,1 & 1,3 & 2,2 \\ \gamma & 2,2 & 3,1 & 1,3 \\ \hline \end{array}$$

The first number of each pair in the matrix gives the ranking of women by the men, the second number is the ranking of the men by the women. Thus, α ranks A first, B second, C third, while A ranks β first, γ second, and α third, etc.

1. What if there is a man and a woman that prefer each other best? What must be true in any matching that hopes to achieve stability?
2. What if the preference lists for the men and woman are all the same?

SOME THOUGHT EXPERIMENTS

- Men propose to women simultaneously in "rounds."
- In round 1, each man proposes to his top woman.
- Each woman evaluates her proposals (if any), and accepts the best, rejecting all others. These women are now "engaged."
- In round 2, each rejected man now proposes (simultaneously) to his second choice; his first choice didn't work out.
- Each woman evaluates her proposals, even if currently engaged, and accepts the best, breaking an engagement if necessary.
- Rounds continue so long as there are still rejected men left to propose, or equivalently until each woman has received a proposal.
- Once each woman is engaged, the mass wedding takes place!

GS "DEFERRED-ACCEPTANCE" OR "PROPOSAL" ALGORITHM

1. It always terminates, in fact in at most (n-1)(n-1)+1 rounds. Why?
2. What about the average/mean/expected number of rounds?
3. It delivers a stable matching. Why?
4. Clearly, men and women can exchange roles, and the women could propose instead. The GS algorithm is "proposer optimal" in the sense that the proposing group simultaneously do as good as each can in any stable matching. Why?
5. It is "proposee floptimal" in the sense that the group being proposed to simultaneously do as badly as each can in any stable matching. Why?

SOME FACTS ABOUT THE GS ALGORITHM (AND SOME HOMEWORK)

Example 2. The ranking matrix is the following.

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline \alpha & 1,3 & 2,3 & 3,2 & 4,3 \\ \beta & 1,4 & 4,1 & 3,3 & 2,2 \\ \gamma & 2,2 & 1,4 & 3,4 & 4,1 \\ \delta & 4,1 & 2,2 & 3,1 & 1,4 \\ \hline \end{array}$$

There is only the one stable set of marriages indicated by the circled entries in the matrix. Note that in this situation no one can get his or her first choice if stability is to be achieved.

- How can we modify the GS algorithm with quotas for colleges that are larger than 1?

BACK TO STUDENTS/COLLEGES

OTHER APPLICATIONS TO THE "STABILITY OF MARRIAGE"?

- Stable Polygamy (multiple wives/husbands)?
- Stable "Roommates" (same gender)?
  - Let's look more closely at this one.

- Suppose there are 4 girls (A, B, C, and D) that need to room together in pairs.
- Suppose A ranks B best, B ranks C best, C ranks A best, and all three rank D worst. What will happen? Do D's rankings matter?

A "STABLE ROOMMATES" PROBLEM

"THE READER WHO HAS FOLLOWED US THIS FAR HAS DOUBTLESS NOTICED A CERTAIN TREND IN OUR DISCUSSION. IN MAKING THE SPECIAL ASSUMPTIONS NEEDED IN ORDER TO ANALYZE OUR PROBLEM MATHEMATICALLY, WE NECESSARILY MOVED FURTHER AWAY FROM THE ORIGINAL COLLEGE ADMISSION QUESTION, AND EVENTUALLY IN DISCUSSING THE MARRIAGE PROBLEM, WE ABANDONED REALITY ALTOGETHER AND ENTERED THE WORLD OF MATHEMATICAL MAKE-BELIEVE. THE PRACTICAL-MINDED READER MAY RIGHTFULLY ASK WHETHER ANY CONTRIBUTION HAS BEEN MADE TOWARD AN ACTUAL SOLUTION OF THE ORIGINAL PROBLEM. EVEN A ROUGH ANSWER TO THIS QUESTION WOULD REQUIRE GOING INTO MATTERS WHICH ARE NONMATHEMATICAL, AND SUCH DISCUSSION WOULD BE OUT OF PLACE IN A JOURNAL OF MATHEMATICS. IT IS OUR OPINION, HOWEVER, THAT SOME OF THE IDEAS INTRODUCED HERE MIGHT USEFULLY BE APPLIED TO CERTAIN PHASES OF THE ADMISSIONS PROBLEM."

~~ GALE & SHAPLEY

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012
Alvin E. Roth, Lloyd S. Shapley

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The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012

Photo: U. Montan

Alvin E. Roth

Prize share: 1/2

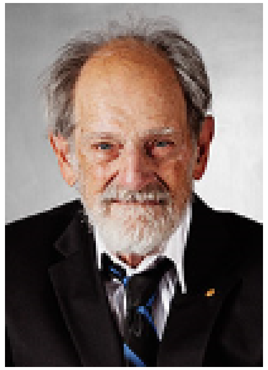


Photo: U. Montan

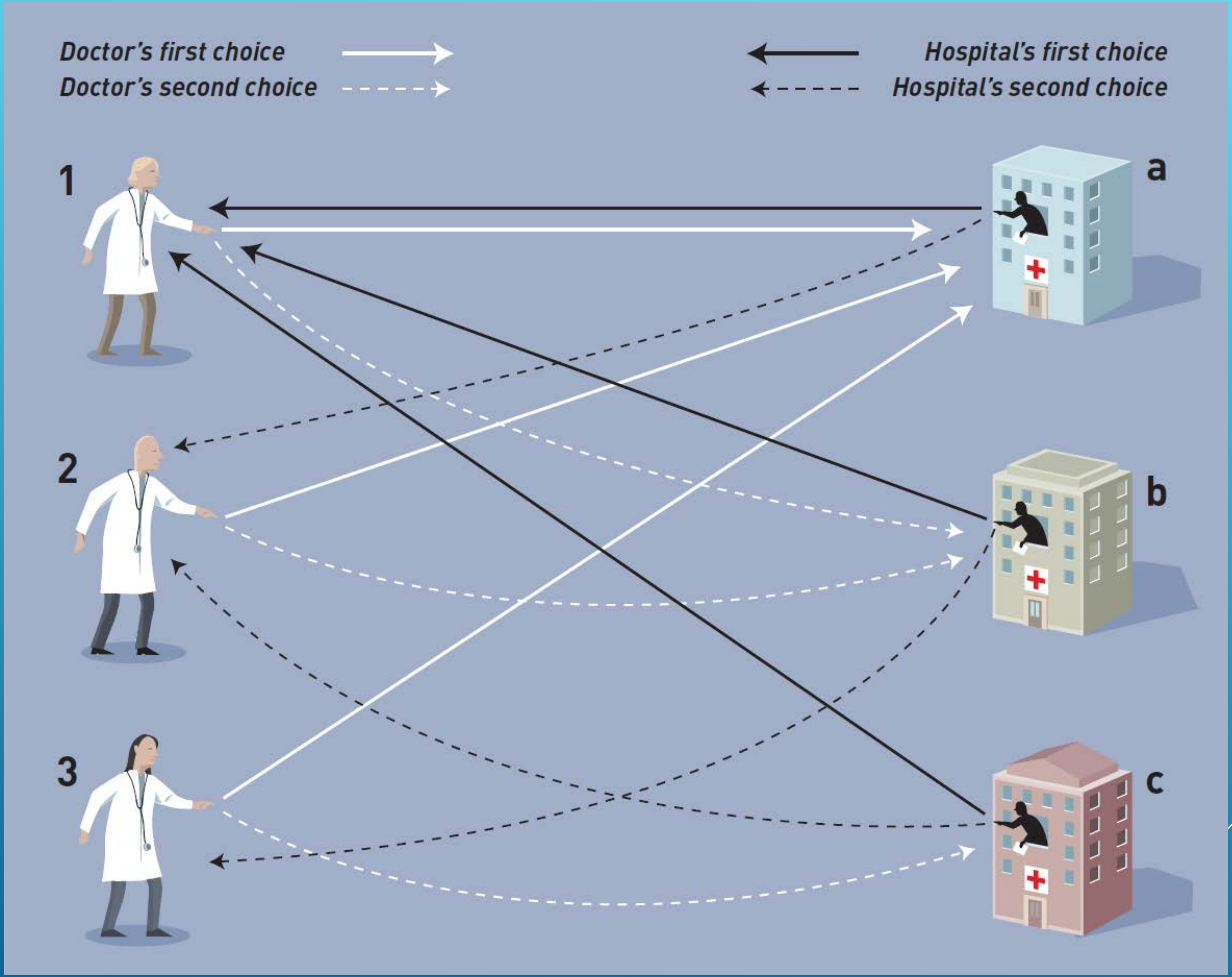
Lloyd S. Shapley

Prize share: 1/2

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012 was awarded jointly to Alvin E. Roth and Lloyd S. Shapley *"for the theory of stable allocations and the practice of market design"*

- Lloyd Shapley had laid the foundation in the abstract realm, starting with his joint work with David Gale in the 1950s and 1960s.
- Alvin Roth had realized, starting in the 1980s, that Shapley's continued work could be adapted and applied in a broad range of practical scenarios, including stable assignments of:
  - new doctors to hospitals (residencies, NRMP);
  - students to schools (school choice); and
  - human organs for transplant to recipients.

GALE & SHAPLEY WERE RIGHT!



Doctor's first choice
Doctor's second choice
Hospital's first choice
Hospital's second choice
1
2
3
a
b
c

- Consider the preference lists to be (uniformly) random, in the *n* men *n* women situation.
- Each collection of preference lists (how many?) is called an "instance of the stable marriage problem."
- Let S be the number of stable matchings in the random instance of the stable marriage problem.
- Know: $\Pr(S \geq 1) = 1$.
- What about the *expected* value of S, E[S]? Can this be computed? If so, is E[S] indicative of a *likely* value for S?

PROBABILITY, ANYONE?

FACTS ABOUT “S”

- Donald Knuth (1976) found an integral formula for E[S].
- Knuth (1978) produced a problem instance that had $2^{n/2}$ stable matchings.
- Robert Irving and Paul Leather (1986) extended Knuth’s example, and algorithmically found problem instances with more than 2^n stable matchings.
- Knuth extended the work of Irving and Leather, and showed that their algorithm produced problem instances with at least 2.28^n stable matchings.
- Open Problem: Is Irving-Leather’s problem instance best possible?

FACTS ABOUT “S”

- Meanwhile, in 1972 McVitie and Wilson had found a sequential version of the GS algorithm, where men propose one-at-a-time.
- In 1990, Knuth, Motwani and Pittel extended the sequential GS algorithm in such a way that it delivered *all possible* stable husbands for any given women, and used this to show that with probability tending to 1 the number of those stable husbands is roughly 0.5 ln n.
- Thus, with high probability, $S \geq 0.5 \ln n$. (!!)
- But later (1992) Boris Pittel showed that, actually, $S \geq (n/\ln n)^{1/2}$ with high probability.
- Can we do better? That is, can we show that S is *even larger* with high probability?

- Boris Pittel (1986) showed that E[S] ~ $e^{-1}n \ln n$ by using Knuth's formula for E[S]. This suggests (but does not prove) that *most* instances of the stable marriage problem have *lots* of stable matchings.
- To prove that the (asymptotic) value of E[S] is actually a *likely* value, Craig Lennon and Boris Pittel (2008) managed to show that $E[S^2] \sim (e^{-2} + 0.5e^{-3})n^2 \ln^2 n$.
- The combination of E[S] and $E[S^2]$ imply (Cantelli's inequality) that at least 84% of stable marriage problem instances have $cn \ln n$ stable matchings!

FACTS ABOUT "S"

- Here, assume that there are *n* people of the same gender, each with their own preference list (having ranked everyone but themselves from best-to-worst). How many?
- Each collection of preference lists represents an “instance of the stable roommates problem.”
- For a (uniformly) random problem instance, let R be the number of stable matchings.
- Know: $\Pr(R \geq 1) < 1$, in contrast with S.
- A problem instance x is said to be “solvable” if $R(x) \geq 1$, so $\Pr(\text{random problem instance } x \text{ is solvable}) = \Pr(R \geq 1)$.

WHAT ABOUT “STABLE ROOMMATES”?

"R" VERSUS "S" SUMMARY

- $cn^{-1/2} \le \Pr(R \ge 1) \le e^{1/2}/2 = 0.82436\ldots$
  - Irving and Pittel
- $E[R] \sim e^{1/2} = 1.64872\ldots$ (Pittel)
- Conjecture (Mertens, 2005):
  - $\Pr(R \ge 1) \sim cn^{-1/4}$

- $\Pr(S \ge 1) = 1$
  - GS algorithm
- $E[S] \sim e^{-1} n \ln n$ (Pittel)
  - … and this is close to a likely value (Lennon and Pittel)

- WONDERFUL reference:
  - Dan Gusfield & Robert W. Irving, *The Stable Marriage Problem: Structure and Algorithms*. The MIT Press, Cambridge, Massachusetts, 1989.

THANK YOU!