An Overview of Monstrous Moonshine

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Abstract

The Conway-Norton monstrous moonshine conjecture set off a quest to discover the connection between the Monster and the J-function. The goal of this poster is to give an overview of the components of the conjecture, the conjecture itself, and some of the ideas that led to its solution. Special focus is given to Klein's J-function.

The Monster Group

In 1973, Fischer and Griess independently conjectured that the Monster group, denoted $\mathbb{M}$, existed. In 1980, Griess constructed it by hand as the automorphism group of a 196,883-dimensional commutative nonassociative algebra [2]. The Monster is the largest of a collection of groups called the sporadic groups. The order of the Monster (the number of elements in the Monster considered as a set) is

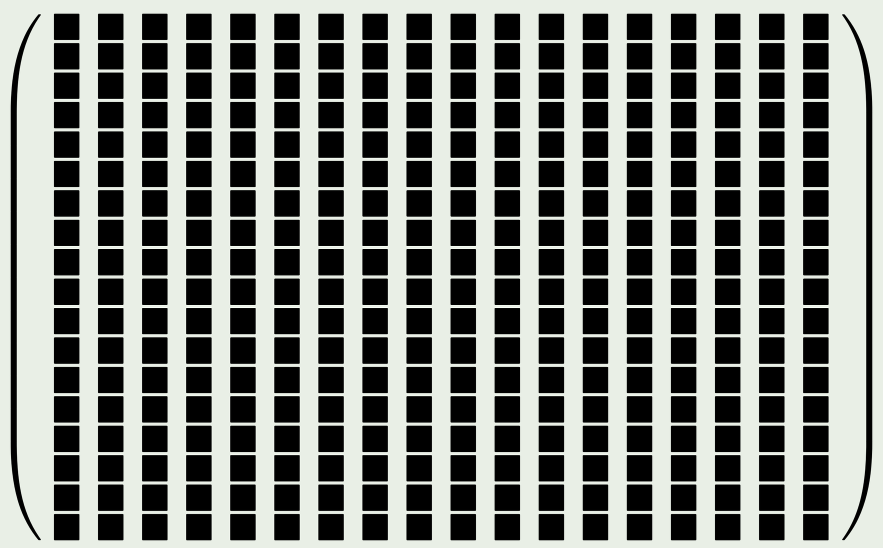
$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \times 10^{53}.$$

We want to think about ways this group can be represented.

Representation and Character Theory

We can consider any finite group G as a finite subgroup of the group $GL(n, \mathbb{C})$ of $n \times n$ invertible matrices with entries that are complex numbers via a special function f which takes each element g in the group G to an n by n matrix in $GL(n, \mathbb{C})$. The function f is called a representation and is a homomorphism. We can break a homomorphism down into multiple homomorphisms. Concretely, since our homomorphism sends the elements of the group to matrices, this looks like breaking down a matrix into smaller matrices along the diagonal [3].

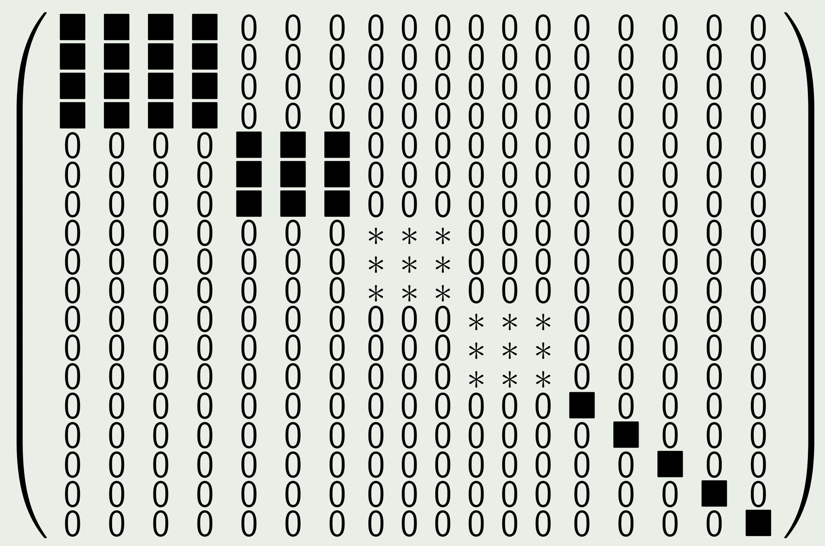
As an example, let us think about A_5, the group of even permutations of a five element set. Suppose each element of A_5 maps to an 18 by 18 matrix. An example of what such a matrix would look like is

,

where the black squares represent elements in $\mathbb{C}$ (the complex numbers).

Given that A_5 has irreducible representations of dimension 5, dimension 4, two different representations of dimension 3, and one of dimension 1, this representation can be broken down into irreducibles, perhaps with one block of dimension 4, one block of dimension 3, two blocks of the other representation of dimension 3, and five blocks of dimension 1. This gives us

Representation and Character Theory (cont.)

,

where the first representation of dimension 3 is shown by the black squares and the other representation is shown by the asterisks.

The Monster has 194 equivalence classes of irreducible representations. In 1978, Fischer, Livingstone, and Thorne determined the dimensions of these representations. Starting with the trivial representation, the dimensions of the first few irreducible representations are

$$(r_n)_{n=1,\cdots,194} = (1, 196\,883, 21\,296\,876, 842\,609\,326, 18\,538\,750\,076, \cdots)$$

and the rest of the sequence can be found in the ATLAS of Finite Groups or the Online Encyclopaedia of Integer Sequences [4].

Preliminary Notation for the J-Function

To understand the J-function, we need to understand elliptic functions. For a function to be elliptic it must be doubly periodic, which means it has two periods, ω_1 and ω_2, both of which are complex numbers, with ratio ω_2/ω_1, where this ratio is not real. A pair of periods (ω_1, ω_2) for a function f is a fundamental pair if all the periods of f are in the set $\Omega = \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}$. We can create a parallelogram by considering $0, \omega_1, \omega_2$, and $\omega_1 + \omega_2$.

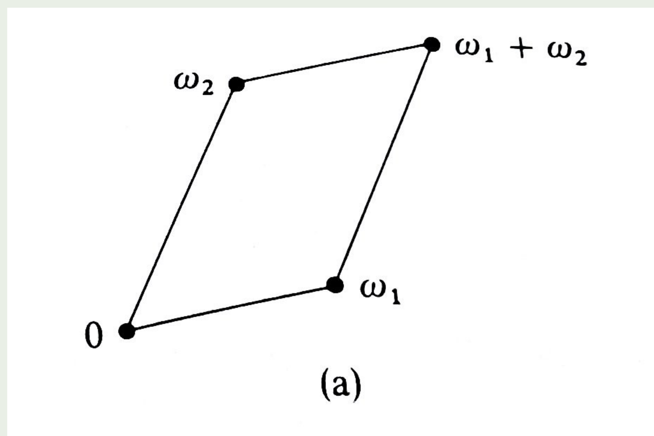
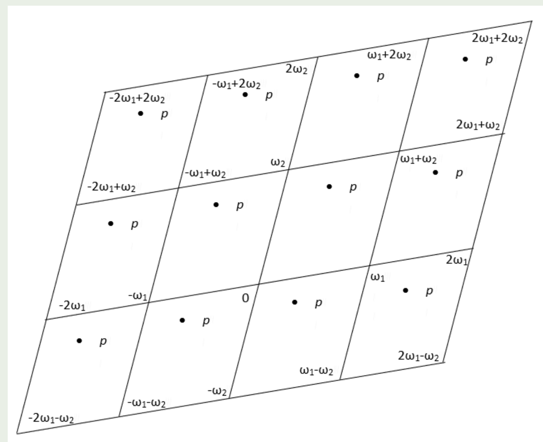


Figure 1.2.a from [1]



More of the Lattice

This parallelogram is important because it is part of a larger lattice. If we understand what is happening in this one parallelogram, for instance, if it has a singularity at p, we will know what is happening in all the other parallelograms in the lattice.

The J-function is an elliptic function defined as quotient of two other functions. We define invariants $g_2 = 60G_4$ and $g_3 = 140G_6$, where $G_n = \sum_{\omega\in\Omega,\ \omega\neq 0} \frac{1}{\omega^n}$. One more function, called the discriminant, is defined by $\Delta = g_2^3 - 27g_3^2$.

Definition of the J-Function

We now define Klein's J-function. This function is a combination of g_2 and Δ. For $\omega_2/\omega_1 \notin \mathbb{R}$ (the real numbers), we define Klein's function as

$$J(\omega_1, \omega_2) = \frac{g_2^3(\omega_1, \omega_2)}{\Delta(\omega_1, \omega_2)} = J(\lambda\omega_1 + \lambda\omega_2)$$

for $\lambda \neq 0$. For $\tau = \omega_2/\omega_1$ we have that $J(1, \tau) = J(\omega_1, \omega_2)$ [1].

The J-function can be written as a Fourier expansion, or a sum in terms of exponentials. Formally, we have the Fourier expansion

$$12^3 J(\tau) = e^{-2\pi i\tau} + 744 + \sum_{n=1}^{\infty} c(n)e^{2\pi in\tau},$$

where the $c(n)$ are integers [1].

The coefficients of this sum have been calculated for $n \leq 100$. The first few are $c(0) = 744$, $c(1) = 196\,884$, $c(2) = 21\,493\,760$, and $c(3) = 864\,299\,970$. In some versions of this formula the 744 is subtracted from both sides, and the result is called $J(\tau)$. For $q = e^{2\pi i\tau}$, that gives us $12^3 J(\tau) - 744 = q^{-1} + 196\,884q + 21\,493\,760q^2 + 864\,299\,970q^3 + \cdots$.

From now on we will refer to this version of the formula as $J(\tau)$.

Monstrous Moonshine

At this point, we return to the Monster. Recall that the first few dimensions of irreducible representations of the Monster are

$$(r_n)_{n=1,\cdots,194} = (1, 196\,883, 21\,296\,876, 842\,609\,326, 18\,538\,750\,076, \cdots).$$

We now notice that

$$\begin{aligned} 196\,884 &= 1 + 196\,883 \\ 21\,493\,760 &= 1 + 196\,883 + 21\,296\,876 \\ 864\,299\,970 &= 2 \cdot 1 + 2 \cdot 196\,883 + 21\,296\,876 + 842\,609\,326, \end{aligned}$$

where the numbers on the left hand side of the equation are coefficients from $J(\tau)$ and the numbers on the right hand side are from the dimensions of the irreducible representations of the Monster. Although Conway and Norton's conjecture that there was a reason behind this connection was first seen as "moonshine", or a crazy idea, in 1992 Borcherds proved that a connection existed using a vertex operator algebra and a result from string theory. So the connection between the J-function and the Monster group was formally shown. Hence, two different areas of mathematics were united in a beautiful way [2].

References

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[4] Valdo Tatitscheff. *A short introduction to Monstrous Moonshine*. 2019. arXiv: 1902.03118 [math.NT].

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