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CONVEXITY PROPERTIES OF HOLOMORPHIC MAPPINGS IN $\mathbb{C}^n$

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Abstract. Not many convex mappings on the unit ball in $\mathbb{C}^n$ for $n > 1$ are known. We introduce two families of mappings, which we believe are actually identical, that both contain the convex mappings. These families which we have named the "Quasi-Convex Mappings, Types A and B" seem to be natural generalizations of the convex mappings in the plane. It is much easier to check whether a function is in one of these classes than to check for convexity. We show that the upper and lower bounds on the growth rate of such mappings is the same as for the convex mappings.

1. Introduction

In the complex plane analytic functions which map the unit disk onto starlike or convex domains have been extensively studied. These functions are easily characterized by simple analytic or geometric conditions and there are many well known results which help us understand their nature. In moving to higher dimensions several difficulties arise. Some are predictable, some are somewhat surprising. Imposing the condition that a mapping be convex turns out to be very restrictive and so we will introduce a larger class of mappings with properties similar to the convex mappings in the plane. We actually look at two classes, the "Quasi-Convex Mappings, Types A and B", but we suspect that they are the same. This paper will contain:

- A brief review of results in the plane with a discussion of some of the difficulties encountered in extending the results to higher dimensions.
- Some characterizations of convex and starlike mappings in higher dimensions.
- The introduction of the "Quasi-Convex" families of mappings in $\mathbb{C}^n$ along with some preliminary results.
- A discussion of open questions.

Before going further let us define some terms which will recur.

- Let X be a Banach space. The ball of radius r, $B_r = \{Z \in X : \|Z\| < r\}$. If $r = 1$, we will simply use B and if $X = \mathbb{C}$, then $B = \Delta$.
- A set A is **convex** if $z, w \in A \Rightarrow tz + (1-t)w \in A$, for all $t \in [0, 1]$, and a mapping is said to be **convex** if it maps the unit ball onto a convex domain.
- A set A is **starlike** with respect to $z_0 \in A$ if $z \in A \Rightarrow (1-t)z + tz_0 \in A$, for all $t \in [0, 1]$. We will use the term **starlike** to mean "starlike with respect to

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0". A mapping is said to be **starlike** if it maps the unit ball onto a starlike domain.

- $S = \{f : \Delta \to \mathbb{C} : f$ is analytic and univalent, $f(0) = 0$ and $f'(0) = 1\}$.
- $S^* = \{f \in S : f(\Delta)$ is starlike with respect to $0\}$.
- $K = \{f \in S : f(\Delta)$ is convex$\}$.

In trying to obtain analogous results in higher dimensions we run into several problems. For example, in proving the result

$$f \in K \Leftrightarrow \operatorname{Re}\left\{\frac{zf''(z)}{f(z)} + 1\right\} > 0,$$

we use the fact that if f is convex, then the tangent vector turns in one direction. *i.e.* $\frac{d}{d\theta}(\arg(izf'(z))) > 0,\ z = re^{i\theta}$. In higher dimensions this concept has no meaning.

Similarly, the characterization,

$$f \in S^* \Leftrightarrow \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$$

is obtained from the observation that for f to be starlike $\frac{d}{d\theta}(\arg f(re^{i\theta})) > 0$. Once again this has no meaning in higher dimensions, nor does the expression $zf'(z)/f(z)$.

The analogue of the well-known equivalence, "$f \in K \Leftrightarrow zf' \in S^*$" is false in higher dimensions as we show in Examples 5 and 8.

Our intuition seems to let us down when we realize that even if we take a function $f \in K$ and form a function $F : B \subset \mathbb{C}^2 \to \mathbb{C}^2$ with $F(z,w) = (f(z), f(w))$, then F is not necessarily convex. This is demonstrated in the following example.

Example 1. Let B be the Euclidean ball in $\mathbb{C}^2$, then the mapping

$$F(z,w) = \left(\frac{z}{1-z}, \frac{w}{1-w}\right), \qquad z, w \in \mathbb{C},\ |z|^2 + |w|^2 < 1,$$

is not convex even though $f(z) = z/(1-z),\ z \in \Delta$, is a convex function in the plane.

Note that $u = f(z)$ maps the real line segment $-1 < z < 1$ onto the real half line $u > -1/2$. A necessary condition that F is convex is that every cross-section of the image of B is convex. Consider the cross-section of $F(B)$, $\{(u,v) \in F(B) : u, v \in \mathbf{R}\}$. This is the image of $\Omega = \{(s,t) \in B : -1 < s < 1,\ -1 < t < 1\}$. This cross-section, $F(\Omega)$, is not convex. If it were, then the set $\{(u,v) : u > 0,\ v > 0\}$ would be contained in $F(\Omega)$. In particular the line $\{(u,u) : u > 0\} \subset F(\Omega)$. If $u = v$, then $s = t$ and $s^2 + t^2 < 1$, $s = t < 1/\sqrt{2}$, so that $u = v < 1/(\sqrt{2} - 1)$. We cannot get any further from the origin along this line and so it is clear that this cross-section is not convex. See Figure 1. Similar arguments show that there is no convex mapping $F(z,w) = (z/(1-z), g(w))$.

In one approach to extending these results to $\mathbb{C}^n$, $n \geq 2$, Suffridge [11] generalizes some of Robertson's results [9], which use the principle of subordination in the plane, to higher dimensions.

To extend these theorems to higher dimensions we first need to adapt the Schwarz Lemma accordingly. There are several ways of doing this (see Harris [4]) but the appropriate one for our purposes is as follows.

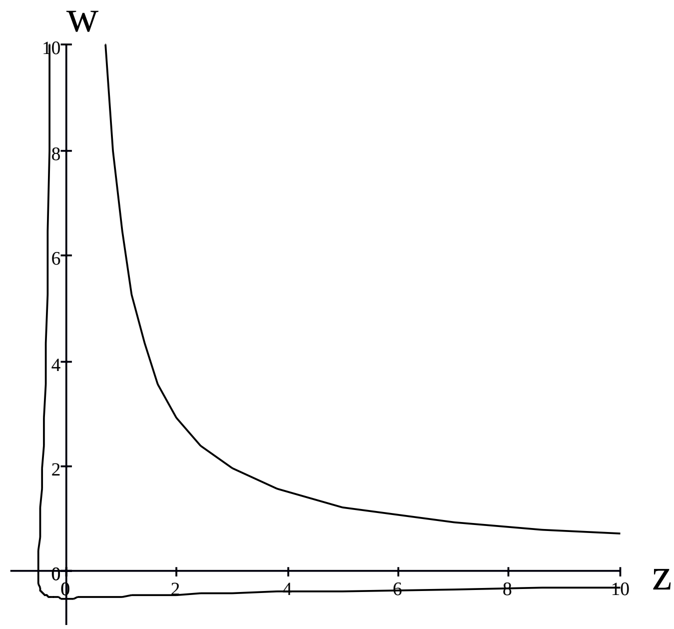

FIGURE 1. $f(z,w) = \left(\frac{z}{1-z}, \frac{w}{1-w}\right)$ for z, w real.

Theorem 1.1. *Let X be a Banach space and let $B \subset X$. If $f : B \to Y$ is holomorphic, $\|f(x)\| \leq 1$ when $x \in B$ and $f(0) = 0$, then $\|f(x)\| \leq \|x\|$ for all $x \in B$.*

We next need to extend the concepts of "positive real part", and "functions of positive real part". We use functionals to accomplish this. For a more complete treatment see Gurganus [3] or Suffridge [14].

Let X be a Banach space and $x \in X$, $x \neq 0$, and let X^* denote the space of linear functionals from X to $\mathbb{C}$. Define

$$T(x) = \{\ell_x \in X^* : \|\ell_x\| = 1, \text{ and } \ell_x(x) = \|x\|\} \text{ where } \|\ell_x\| = \sup_{\|y\|=1} |\ell_x(y)|.$$

Let $H_1 = \{y \in X : \operatorname{Re} \ell_x(y) = \|x\|\}$. H_1 is a supporting hyperplane for $B_{\|x\|}$ at x, because $x \in H_1$ and if $y \in H_1$, then

$$\|y\| \geq |\ell_x(y)| \geq \operatorname{Re}\{\ell_x(y)\} = \|x\|, \text{ since } \|\ell_x\| = 1.$$

If X has complex dimension n, then H_1 has real dimension $2n-1$ and it is thus a hyperplane. In the infinite dimensional case, H_1 has real codimension 1.

Example 2. Let $X = \mathbb{C}^n$ with a p-norm, $1 < p < \infty$, *i.e.* $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$. Then $\ell_x \in T(x)$ is given uniquely by

$$\ell_x(w) = \frac{\sum_{\{j:x_j \neq 0\}} w_j \bar{x}_j |x_j|^{p-2}}{\|x\|_p^{p-1}}.$$

Example 3. In the case of $p = 1$, $T(x)$ is the set of linear functionals of the form

$$\ell_x(w) = \sum_{\{j:x_j \neq 0\}} \frac{w_j \bar{x}_j}{|x_j|} + \sum_{\{j:x_j = 0\}} \gamma_j w_j \text{ with } \gamma_j \in \mathbb{C},\ |\gamma_j| \leq 1 \text{ for all } j.$$

And in the case $p = \infty$, we let $J = \{j : \|x\| = |x_j|\}$ and

$$\ell_x(w) = \sum_{j \in J} \frac{t_j w_j \bar{x}_j}{\|x\|} \text{ where each } t_j \geq 0 \text{ and } \sum_{j \in J} t_j = 1.$$

We now define three families of functions:

$$\begin{aligned} N_0 = \{w : B \to X : w \text{ is holomorphic, } w(0) = 0, \text{ and } \operatorname{Re}\{\ell_x(w(x))\} \geq 0,\\ \text{for all } x \in B,\ x \neq 0,\ \ell_x \in T(x)\},\\ N = \{w \in N_0 : \operatorname{Re}\{\ell_x(w(x))\} > 0, \text{ for all } x \in B,\ x \neq 0,\ \ell_x \in T(x)\},\\ M = \{w \in N : Dw(0) = I\}. \end{aligned}$$

Example 4. Let $X = \mathbb{C},\ B = \Delta$, then

$$N_0 = \{w : \Delta \to \mathbb{C} : w \text{ is analytic, } w(0) = 0,\ \operatorname{Re}\{\bar{z}w(z)\} \geq 0,\ z \in \Delta \backslash \{0\}\}.$$

However ,

$$\begin{aligned} \operatorname{Re}\{\bar{z}w(z)\} \geq 0 \quad &\Leftrightarrow \quad \operatorname{Re}\left\{|z|^2 \frac{w(z)}{z}\right\} \geq 0\\ &\Leftrightarrow \quad \operatorname{Re}\left\{\frac{w(z)}{z}\right\} \geq 0. \end{aligned}$$

Thus, if $w \in N_0$ either $\operatorname{Re}\{w(z)/z\} \equiv 0$ or $\operatorname{Re}\{w(z)/z\} > 0$.

We also observe that $M = \{zf : f \in \mathbb{P}\}$, where $\mathbb{P}$ is the family of functions that are analytic in the unit disk with $f(0) = 1$ and $f(\Delta)$, contained in the right half-space.

The following lemmas are Suffridge's extensions of Robertson's theorems, [14, 11].

Lemma 1 (Suffridge)**.** *Let $v : B \times I \to B$ be holomorphic in B for each $t \in I = [0, 1]$ (i.e. $v(\cdot, t)$ is holomorphic for each fixed $t \in I$), $v(0, t) = 0$ and $v(x, 0) = x$. If $\lim_{t \to 0^+} \frac{x - v(x,t)}{t} = w(x)$ exists and is holomorphic in B, then $w \in N_0$.*

Lemma 2 (Suffridge)**.** *Let $f : B \to Y$ be a biholomorphic mapping of B onto an open set $f(B) \subset Y$ and let $f(0) = 0$. Assume $F : B \times I \to Y$ is holomorphic in B for each fixed $t \in I$, $F(x, 0) = f(x)$, $F(0, t) = 0$ and suppose $F(B, t) \subset f(B)$ for each fixed $t \in I$. Further, suppose*

$$\lim_{t \to 0^+} \frac{F(x, 0) - F(x, t)}{t} = G(x)$$

exists and is holomorphic. Then $G(x) = DF(x)(w(x))$ where $w \in N_0$.

From these we obtain the characterization of starlike mappings in higher dimensions. Note the similarity of this condition to that of starlike functions in the plane.

Theorem 1.2. *The mapping $f : B \subset \mathbb{C}^n \to \mathbb{C}^n$ is starlike if and only if $f(x) = Df(x)(\omega(x))$ for some $\omega \in M$.*

This result was obtained by Matsuno [6] for the Euclidean norm, and by Suffridge for the sup norm [11], and for more general norms [13].

We include two examples of mappings which are starlike.

Example 5. The function $f(Z) = (z + aw^2, w)$ where $Z = (z, w)$, $\|Z\|^p = |z|^p + |w|^p < 1$, $p > 1$, $z, w \in \mathbf{C}$ is starlike if and only if

$$|a| \le \left(\frac{p^2-1}{4}\right)^{1/p} \left(\frac{p+1}{p-1}\right). \tag{1}$$

We have that f is starlike if and only if $\operatorname{Re}\left\{\ell_Z(Df(Z)^{-1}(f(Z)))\right\} > 0$. Since

$$Df(Z)^{-1}(f(Z)) = \begin{bmatrix} z - aw^2 \\ w \end{bmatrix},$$

we have

$$\operatorname{Re} \ell_Z(Df(Z)^{-1}(f(Z))) = \operatorname{Re}\left\{\|Z\|^p - aw^2\bar{z}|z|^{p-2}\right\}/\|Z\|^{p-1}.$$

Replacing Z by αZ, $|\alpha| < 1/\|Z\|$, we apply the minimum principle for harmonic functions to see that we may assume that $\|Z\| = 1$. Thus the necessary and sufficient condition for f to be starlike is

$$\begin{aligned} \operatorname{Re}\left\{1 - aw^2\bar{z}|z|^{p-2}\right\} &\ge 1 - |a||w|^2|z|^{p-1} \\ &= 1 - |a|(1-r^p)^{2/p} r^{p-1}, \text{ where } r = |z|. \end{aligned}$$

By elementary calculus, write $h(r) = 1 - |a|(1-r^p)^{2/p} r^{p-1}$, $h(r)$ has

$$1 - |a| \left(\frac{2}{p+1}\right)^{2/p} \left(\frac{p-1}{p+1}\right) \left(\frac{p-1}{p+1}\right)^{-1/p}$$

as its minimum value and (1) follows. □

Note that this result together with Example 7 tells us that the result "$Df(Z)(Z)$ is starlike implies f is convex " does not hold for $n > 1$. If we use the 2-norm and let $f(z, w) = (z + \frac{a}{2}w^2, w)$, then $Df(z, w)(z, w) = (z + aw^2, w)$ and this is starlike for $|a| \le 3\sqrt{3}/2$. However, f is only convex for $|a| \le 1$.

Example 6. The mapping $f : B \subset \mathbb{C}^2 \to \mathbb{C}^2$ given by $f(z, w) = (z + azw, w)$, with $|z|^p + |w|^p < 1$ is starlike if and only if $|a| \le 1$ for all p-norms, $1 \le p \le \infty$.

First assume $1 < p < \infty$, then f is starlike if and only if

$$\operatorname{Re}\left\{\ell_Z(Df(Z)^{-1}(f(Z)))\right\} > 0$$

for $\ell_Z \in T(Z)$.

$$\operatorname{Re}\left\{\ell_Z(Df(Z)^{-1}(f(Z)))\right\} = \operatorname{Re}\left\{\ell_Z\left(\left(\frac{z}{1+aw}, w\right)\right)\right\}.$$

If we use a p-norm and assume by the minimum principle (as before) that $\|Z\| = 1$, we find that

$$\begin{aligned}
\operatorname{Re}\left\{\ell_Z\left(\left(\frac{z}{1+aw}, w\right)\right)\right\} &\geq \operatorname{Re}\left\{\frac{|z|^p}{1+aw} + |w|^p\right\} \\
&= \operatorname{Re}\left\{\frac{|z|^p + |w|^p + aw|w|^p}{1+aw}\right\} \\
&= \operatorname{Re}\left\{\frac{1 + aw|w|^p}{1+aw}\right\} \\
&= \operatorname{Re}\left\{\frac{1 + aw|w|^p + \overline{aw} + |a|^2|w|^{p+2}}{|1+aw|^2}\right\}.
\end{aligned}$$

Clearly $|a| \leq 1$ is necessary and we need to find a such that

$$\operatorname{Re}\left\{1 + |a|^2|w|^{p+2} + \overline{aw} + aw|w|^p\right\} \geq 0.$$

It is sufficient to have

$$1 + |a|^2|w|^{p+2} - |a||w| - |a||w|^{p+1} = (1 - |a||w|^p)(1 - |a||w|^{p+1}) \geq 0$$

and hence the result readily follows. The cases $p = 1$ and $p = \infty$ are easily handled. □

The last theorem might lead us to conjecture that

$$Df(x)^{-1}(D^2 f(x)(x,x) + Df(x)(x)) \in M \text{ if and only if } f \text{ is convex.}$$

The mapping given in Example 1 quickly dispels this thought. It turns out that this is a necessary but not sufficient condition. We will look more extensively at this condition later on. For necessary and sufficient conditions we have the following theorem, [11].

Theorem 1.3 (Suffridge)**.** *Let X and Y be Banach spaces with $B \subset X$. Let $f : B \to Y$ be locally biholomorphic with $f(x) - f(y) = Df(x)(\omega(x,y))$ for $x, y \in B$. Then f is convex if and only if* $\operatorname{Re}\{\ell_x(\omega(x,y))\} > 0$ *whenever $\|y\| < \|x\|$ and $\ell_x \in T(x)$.*

The condition says that $f(B)$ must be starlike with respect to each of its interior points. However, this condition, which agrees with our intuition, is difficult to apply. For a somewhat different approach, see [2, Theorem 2]. The following examples make use of the above theorem.

Example 7. The function $f(Z) = (z + aw^2, w)$ where $Z = (z, w)$ with $\|Z\|^2 = |z|^2 + |w|^2 < 1$ and $z, w \in \mathbf{C}$ is convex if and only if $|a| \leq 1/2$.

We will need to check when $\operatorname{Re}\langle Df(Z)^{-1}(f(Z)-f(U), Z\rangle > 0$ where $Z=(z,w)$ and $U=(u,v)$ with $\|Z\| \geq \|U\|$.

$$\begin{aligned}
&\operatorname{Re}\langle Df(Z)^{-1}(f(Z)-f(U)), Z\rangle \\
&= \operatorname{Re}\{|z|^2+|w|^2-u\bar{z}-v\bar{w}-a\bar{z}(w-v)^2\} \\
&= \|Z\|^2-\operatorname{Re}\langle U,Z\rangle-\operatorname{Re}\{a\bar{z}(w-v)^2\} \\
&\geq \|Z\|^2-\operatorname{Re}\langle U,Z\rangle-|a||z||w-v|^2 \\
&= \|Z\|^2-\operatorname{Re}\langle U,Z\rangle \\
&\quad -|a||z|(\|Z\|^2-|z|^2-2\operatorname{Re}(u\bar{z}+v\bar{w})+2\operatorname{Re}u\bar{z}+\|U\|^2-|u|^2) \\
&= \|Z\|^2(1-|a||z|)-\operatorname{Re}\langle U,Z\rangle(1-2|a||z|)-|a||z|(\|U\|^2-|z-u|^2) \\
&\geq \|Z\|^2(1-|a||z|)-\operatorname{Re}\langle U,Z\rangle(1-2|a||z|)-|a||z|(\|Z\|^2-|z-u|^2) \\
&= (\|Z\|^2-\operatorname{Re}\langle U,Z\rangle)(1-2|a||z|)+|a||z||z-u|^2 \\
&\geq 0 \text{ when } |a| \leq \frac{1}{2},
\end{aligned}$$

since $|\operatorname{Re}\langle U,Z\rangle| \leq \|Z\|^2$, and $|a| \leq \frac{1}{2} \Rightarrow |a||z| \leq \frac{1}{2}$.

If $|a| > \frac{1}{2}$ we can find z such that $\bar{z}a > \frac{1}{2},\ u=z,\ v=-w \in \mathbf{R}$ and we obtain

$$\begin{aligned}
&\operatorname{Re}\langle Df(Z)^{-1}(f(Z)-f(U)), Z\rangle \\
&= \|Z\|^2-\operatorname{Re}\langle U,Z\rangle-\operatorname{Re}\{a\bar{z}(w-v)^2\} \\
&< \|Z\|^2-\operatorname{Re}\{z\bar{z}-w\bar{w}\}-\frac{1}{2}(2w)^2 \\
&= 0. \quad \square
\end{aligned}$$

Example 8. The mapping $f: B \subset \mathbb{C}^2 \to \mathbb{C}^2$, with the 2-norm, given by $f(z,w) = (z+azw, w)$ is convex if and only if $|a| \leq 1/\sqrt{2}$.

It is sufficient to assume that $a>0$. Using the above result we let $Z=(z,w)$ and $U=(u,v)$. Then

$$\begin{aligned}
&\langle Df(Z)^{-1}(f(Z)-f(U)), Z\rangle \\
&= \bar{z}(z-u)\left(\frac{1+av}{1+aw}\right)+\bar{w}(w-v) \\
&= \bar{z}(z-u)\left(1-\frac{a(w-v)}{1+aw}\right)+\bar{w}(w-v) \\
&= \langle Z-U,Z\rangle-\frac{a\bar{z}}{1+aw}(z-u)(w-v).
\end{aligned}$$

So,

$$\begin{aligned}
&\operatorname{Re}\left\{\langle Df(Z)^{-1}(f(Z)-f(U)), Z\rangle\right\} \\
&= \operatorname{Re}\left\{\langle Z-U,Z\rangle-\frac{a\bar{z}}{1+aw}(z-u)(w-v)\right\} \\
&\geq \operatorname{Re}\langle Z-U,Z\rangle-\frac{a|z|}{1-a|w|}|z-u||w-v|.
\end{aligned}$$

By examining the function $\frac{ax}{1-ay}$ subject to the constraint $x^2+y^2=k^2$ we see that $\frac{a|z|}{1-a|w|}$ is maximized at $\frac{a\|Z\|}{\sqrt{1-a^2\|Z\|^2}}$ when $|z| = \|Z\|\sqrt{1-a^2\|Z\|^2}$ and $|w| = a\|Z\|^2$.

Similarly, by maximizing the product xy subject to the constraint $x^2+y^2=k^2$ we see that $|z-u||w-v| \le \frac{1}{2}\|Z-U\|^2$ with equality when $|z-u|=|w-v|$.

Now we have the sharp inequality

$$\begin{aligned} &\operatorname{Re}\{\langle Df(Z)^{-1}(f(Z)-f(U)), Z\rangle\} \\ &\qquad \ge \operatorname{Re}\langle Z-U, Z\rangle - \frac{a\|Z\|}{\sqrt{1-a^2\|Z\|^2}}\frac{1}{2}\|Z-U\|^2. \end{aligned} \tag{2}$$

For $a\|Z\| = 1/\sqrt{2}$ this expression is positive for $\|U\| \le \|Z\|$ since

$$\begin{aligned} &\operatorname{Re}\langle Z-U, Z\rangle - \frac{a\|Z\|}{\sqrt{1-a^2\|Z\|^2}}\frac{1}{2}\|Z-U\|^2 \\ &= \operatorname{Re}\left\{\langle Z-U, Z\rangle - \frac{1}{2}\langle Z-U, Z-U\rangle\right\} \\ &= \|Z\|^2 - \|U\|^2 \ge 0. \end{aligned}$$

Since the function $\dfrac{x}{\sqrt{1-x^2}}$ is increasing on $[0,1)$ the inequality holds for $a\|Z\| \le 1/\sqrt{2}$.

To show that $\operatorname{Re}\left\{\langle Df(Z)^{-1}(f(Z)-f(U)), Z\rangle\right\} < 0$ for $a\|Z\| > 1/\sqrt{2}$ we choose $Z=(z,w)=(k\sqrt{1-a^2k^2},\ -ak^2)$ and

$$U=(u,v)=(z-(z+w)\cos\theta e^{i\theta},\ w-(z+w)\cos\theta e^{-i\theta}),$$

where $0\le k\le 1$ and $\cos\theta \ne 0$. We note that $\|Z\|=\|U\|=k$.

$$\begin{aligned} &\operatorname{Re}\left\{\langle Z-U, Z\rangle - \frac{a\bar{z}}{1+aw}(z-u)(w-v)\right\} \\ &= \frac{(z+w)^2\cos^2\theta}{\sqrt{1-a^2k^2}}(\sqrt{1-a^2k^2}-ak) \\ &< 0, \text{ for } ak > 1/\sqrt{2}. \end{aligned}$$

So, on $B=\{Z : \|Z\|<1\}$ we have $a \le 1/\sqrt{2}$ for convexity. □

In the plane, $f \in K$ if and only if $zf' \in S^*$. This last result shows us that this is not true in higher dimensions. The mapping $Df(Z)(Z) = (z+2azw, w)$, for $a=1/\sqrt{2}$, is not even univalent much less starlike. To see this we note that $(z(1+\sqrt{2}w), w) = (0, -1/\sqrt{2})$ for all $Z=(z,-1/\sqrt{2}),\ Z\in B$.

When we couple this with Example 5 we see that the implication does not hold in either direction.

2. A sufficient condition for convexity

The nature of convex mappings is strongly dependent on the norm used in the domain. Using the sup norm in $\mathbb{C}^n$ so that the unit ball is a polydisk, the only normalized convex mappings are mappings F such that F_j is a function of z_j only and F_j is a convex mapping on the unit disk. On the other hand, using the 1-norm, $\|Z\| = \sum_{j=1}^n |z_j|$, the convex maps of the unit ball are the non-singular linear mappings. In the Euclidean norm (*i.e.* using the 2-norm in $\mathbb{C}^n$) we have the following theorem. In view of the results stated above for the sup norm and the 1-norm, such a result cannot hold for normed linear spaces in general.

Theorem 2.1. *Let $B = \{z \in \mathbb{C}^n : \|z\|^2 = \sum_{i=1}^n |z_i|^2 < 1\}$ and assume $f : B \to \mathbb{C}^n$ is holomorphic with $f(0) = 0$ and $Df(0) = 1$. Further, assume $\sum_{k=2}^{\infty} \frac{k^2}{k!} \|D^k f(0)\| \leq 1$. Then $f(B)$ is convex.*

Proof. Consider a function $A_k : \prod_{j=1}^k \mathbb{C}^n \to \mathbb{C}^n$ that is linear in each variable and symmetric. Then $A_k(z, z, ..., z) \equiv A_k(z^k)$ is a homogeneous polynomial of degree k and by a result of Hörmander [5, Theorem 4], we have,

$$\begin{aligned} \|A_k\| &= \sup_{\substack{\|z^{(j)}\|=1 \\ 1\leq j\leq k}} \|A_k(z^{(1)}, z^{(2)}, ..., z^{(k)})\| \\ &= \sup_{\|z\|=1} \|A_k(z, z, ..., z)\|. \end{aligned}$$

Further, by Lemma 1 in Hörmander's paper above, given $f : B \to \mathbb{C}^n$, where f is holomorphic on the unit ball B of $\mathbb{C}^n$ with k^{th} derivative at 0, $D^k f(0)$. We may identify $\frac{1}{k!} D^k f(0)$ with A_k above. Then

$$f(z) = f(0) + \sum_{k=1}^{\infty} \frac{1}{k!} D^k f(0)(z^k) = f(0) + \sum_{k=1}^{\infty} A_k(z^k).$$

Assuming $f : B \to \mathbb{C}^n$ satisfies $f(0) = 0$, $Df(0) = A_1 = I$ and that

$$\sum_{k=2}^{\infty} k^2 \|A_k\| \leq 1,$$

we proceed as follows.

First observe that

$$\sum_{k=2}^{\infty} k\|A_k\| \leq \frac{1}{2} \sum_{k=2}^{\infty} k^2 \|A_k\| \leq \frac{1}{2}$$

with equality in the first step if and only if $A_k \equiv 0$ when $k > 2$. We also note that

$$\begin{aligned} \left\| \sum_{k=2}^{\infty} k A_k(z^{k-1}, w) \right\| &\leq \sum_{k=2}^{\infty} k\|A_k\| \, \|z\|^{k-1} \|w\| \\ &\leq \|z\| \, \|w\| \sum_{k=2}^{\infty} k\|A_k\| \\ &= N\|z\| \, \|w\| \end{aligned}$$

where

$$N = \sum_{k=2}^{\infty} k\|A_k\| \leq \frac{1}{2} \text{ and } \|z\| \leq 1, \ \|w\| \leq 1.$$

Therefore, it follows that

$$\left\| \left(\sum_{k=2}^{\infty} kA_k(z^{k-1}, \cdot) \right)^p (w) \right\| \leq N^p \|z\|^p \|w\|$$

when p is a non-negative integer.

The analytic condition for f to be convex is that

$$\operatorname{Re}\left\{ \langle Df(z)^{-1}(f(z) - f(w)), z \rangle \right\} > 0 \text{ when } 1 > \|z\| \geq \|w\|.$$

We have

$$Df(z)(u) = \lim_{h \to 0} \frac{f(z + hu) - f(z)}{h} = \sum_{k=1}^{\infty} kA_k(z^{k-1}, u).$$

That is,

$$\begin{aligned} \frac{f(z+hu) - f(z)}{h} &= \sum_{k=1}^{\infty} \sum_{l=1}^{k} \binom{k}{l} h^{l-1} A_k(z^{k-1}, u^l) \\ &\to \sum_{k=1}^{\infty} kA_k(z^{k-1}, u) \text{ as } h \to 0. \end{aligned}$$

Therefore,

$$Df(z)^{-1} = [I - (I - Df(z))]^{-1} \tag{3}$$

$$= \left[I - \sum_{k=2}^{\infty} -kA_k(z^{k-1}, \cdot) \right]^{-1} \tag{4}$$

$$= I + \sum_{l=1}^{\infty} (-1)^l \left(\sum_{k=2}^{\infty} kA_k(z^{k-1}, \cdot) \right)^l, \tag{5}$$

because

$$\left\| \sum_{k=2}^{\infty} kA_k(z^{k-1}, \cdot) \right\| \leq N\|z\| \leq \frac{1}{2} \tag{6}$$

for each fixed z, $\|z\| < 1$.

Also,

$$\begin{aligned} f(z) - f(w) &= \sum_{k=1}^{\infty} [A_k(z^k) - A_k(w^k)] \\ &= \sum_{k=1}^{\infty} \sum_{p=1}^{k} A_k(z^{k-p},\ w^{p-1}, z - w) \\ &= z - w + \sum_{k=2}^{\infty} \sum_{p=1}^{k} A_k(z^{k-p}, w^{p-1}, z - w). \end{aligned}$$

Therefore,

$$
\begin{aligned}
&H(z,w)\\
&\equiv\ Df(z)^{-1}(f(z)-f(w))\\
&=\ \left(\left(I+\sum_{l=1}^{\infty}(-1)^l\left(\sum_{k=2}^{\infty}kA_k(z^{k-1},\cdot)\right)^l\right)\right.\\
&\qquad\left.\times\left(z-w+\sum_{p=2}^{\infty}\sum_{q=1}^{p}A_p(z^{p-q},w^{q-1},z-w)\right)\right)\\
&=\ z-w+\sum_{l=1}^{\infty}(-1)^l\left(\sum_{k=2}^{\infty}kA_k(z^{k-1},\cdot)\right)^l(z-w)\\
&\quad+\sum_{l=1}^{\infty}(-1)^l\left(\sum_{k=2}^{\infty}kA_k(z^{k-1},\cdot)\right)^{l-1}\left(-\sum_{p=2}^{\infty}\sum_{q=1}^{p}A_p(z^{p-q},w^{q-1},z-w)\right)\\
&=\ (z-w)+\sum_{l=1}^{\infty}(-1)^l\left(\sum_{k=2}^{\infty}kA_k(z^{k-1},\cdot)\right)^{l-1}\\
&\quad\left(\sum_{p=2}^{\infty}\left(\sum_{q=1}^{p}A_p(z^{p-1},z-w)-A_p(z^{p-q},w^{q-1},z-w)\right)\right)\\
&=\ z-w+\sum_{l=1}^{\infty}(-1)^l\left(\sum_{k=2}^{\infty}kA_k(z^{k-1},\cdot)\right)^{l-1}\\
&\quad\times\left(\sum_{k=2}^{\infty}\sum_{q=2}^{k}\sum_{p=2}^{q}(A_k(z^{k-p},w^{p-2},(z-w)^2))\right)
\end{aligned}
$$

Now

$$
\begin{aligned}
&\left\|\sum_{k=2}^{\infty}\sum_{q=2}^{k}\sum_{p=2}^{q}A_k(z^{k-p},w^{p-2},(z-w)^2)\right\|\\
&\quad\le\ \sum_{k=2}^{\infty}\sum_{q=2}^{k}\sum_{p=2}^{q}\|A_k\|\,\|z\|^{k-p}\|w\|^{p-2}\|z-w\|^2\\
&\quad\le\ \sum_{k=2}^{\infty}\sum_{q=2}^{k}\sum_{p=2}^{q}\|A_k\|\,\|z-w\|^2=\sum_{k=2}^{\infty}\sum_{q=2}^{k}(q-1)\|A_k\|\,\|z-w\|^2\\
&\quad=\ \sum_{k=2}^{\infty}\frac{(k-1)(k)}{2}\|A_k\|\,\|z-w\|^2\\
&\quad\le\ \frac{1-N}{2}\|z-w\|^2.
\end{aligned}
$$

Now assume $\|w\|\le\|z\|=r<1$. Then

$$
\begin{aligned}
\frac{\|z-w\|^2}{2}\ &=\ \frac{1}{2}(\|z\|^2-2\operatorname{Re}\langle w,z\rangle+\|w\|^2)\\
&\le\ r^2-\operatorname{Re}\langle w,z\rangle>0\text{ if }w\ne z.
\end{aligned}
$$

Thus,

$$\mathrm{Re}\langle H(z,w),z\rangle = \mathrm{Re}\left\{\langle z-w,z\rangle + \langle H(z,w)-(z-w),z\rangle\right\}$$

while

$$\begin{aligned}
\|H(z,w)-(z-w)\| &\leq \sum_{l=1}^{\infty} \|\sum_{k=2}^{\infty} kA_k(z^{k-1},\cdot)\|^{l-1} \\
((1-N)(r^2-\mathrm{Re}\,\langle w,z\rangle)) &\leq \sum_{l=1}^{\infty} (N\|z\|)^{l-1}(1-N)(r^2-\mathrm{Re}\langle w,z\rangle \\
&= \frac{1-N}{1-N\|z\|}\,(r^2-\mathrm{Re}\,\langle w,z\rangle)
\end{aligned}$$

Since $\langle z-w,z\rangle = r^2-\langle w,z\rangle \geq 0$ we have

$$\begin{aligned}
\mathrm{Re}\,\langle H(z,w),z\rangle &\geq (r^2-\mathrm{Re}\,\langle w,z\rangle)\left(1-\frac{1-N}{1-N\|z\|}\|z\|\right) \\
&= (r^2-\mathrm{Re}\,\langle w,z\rangle)\left(\frac{1-\|z\|}{1-r\|z\|}\right) \geq 0
\end{aligned}$$

and the proof is complete. □

3. The quasi-convex mappings

As we have seen, the condition that a mapping be convex is somewhat restrictive and unwieldy to verify. You will recall that even the mapping $(f_1(z_1),\ldots,f_n(z_n))$ with $f_j:\Delta\to\mathbb{C}$ convex for each $j=1,\ldots,n$, may not be convex in $\mathbb{C}^n$. This leads us to consider a set of mappings which contains the set of convex mappings for dimension two or more and has many of the "nice" properties that we would like a generalization of the convex functions in the plane to have, yet has a more readily usable definition.

The characterization

$$f\in K \text{ if and only if } \mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}+1\right\}>0 \tag{7}$$

is well-known and as we have mentioned comes from the fact that the curvature of the boundary of the image of any disk $|z|<r<1$ is always positive if and only if the function is convex. A less well-known result is that (see Suffridge, [12]),

$$f\in K \text{ if and only if } \mathrm{Re}\left\{\frac{zf'(z)}{f(z)-f(\xi)}\right\}>0, \text{ for all } z,\xi\in\Delta,\ |\xi|<|z|. \tag{8}$$

This characterization comes from noticing that f being convex is equivalent to f being starlike with respect to every interior point. The expression is arrived at by letting z vary on a circle of radius r and then for any fixed ξ with $|\xi|<r<1$, the argument of the vector connecting $f(\xi)$ with $f(z)$ is an increasing function of $\arg(z)$.

If $|\xi|=r,\ \xi\neq z$, then from (8) we have

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)-f(\xi)}\right\}\geq 0, \tag{9}$$

We further note that when $|z| = |\xi|$, $z \neq \xi$, $\mathrm{Re}\left\{\frac{z+\xi}{z-\xi}\right\} = 0$. Hence

$$\mathrm{Re}\left\{\frac{2zf'(z)}{f(z)-f(\xi)} - \frac{z+\xi}{z-\xi}\right\} \geq 0. \tag{10}$$

We observe that the singularity at $z = \xi$ is removable and we are now working with the real part of an analytic function of z and ξ, which is thus harmonic in both z and ξ.

By fixing z and varying ξ, since we know that this function cannot attain its minimum on the interior of the disk $|\xi| < r$, the inequality is strict on the interior. Similarly, by holding ξ fixed and varying z we get the same result for z. We conclude that

$$f \in K \text{ if and only if } \mathrm{Re}\left\{\frac{2zf'(z)}{f(z)-f(\xi)} - \frac{z+\xi}{z-\xi}\right\} \geq 0, \text{ for all } z, \xi \in \Delta. \tag{11}$$

We further note that

$$\lim_{\xi \to z} \mathrm{Re}\left\{\frac{2zf'(z)}{f(z)-f(\xi)} - \frac{z+\xi}{z-\xi}\right\} = \mathrm{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\}. \tag{12}$$

When trying to generalize these ideas, we have seen in Theorem 1.3 that (8) does extend to higher dimensions. However, in trying to generalize the expression in (11) we find that we cannot find an appropriate second term that removes the singularity. So we need to modify the approach.

Definition 1. Let

$$\mathbb{S}_n = \{f : B \subset \mathbb{C}^n \to \mathbb{C}^n : f(0) = 0 \text{ and } Df(0) = I\},$$

and let

$$S^{2n-1} = \{U \in \mathbb{C}^n : \|U\| = 1\}.$$

represent the unit sphere in $\mathbb{C}^n$.

Consider the one-dimensional subset of B,

$$C_U = \{\alpha U : U \in S^{2n-1}, U \text{ fixed, and } \alpha \in \Delta\}.$$

On this slice of B we can mimic the expression in (11) in the following way.

Definition 2. Let $U \in \mathbb{C}^n$, with $\|U\| = 1$, and let $\ell_U \in T(U)$. For $f \in \mathbb{S}_n$ define $G_f : \Delta \times \Delta \to \widehat{\mathbb{C}}$ by

$$G_f(\alpha, \beta) = \frac{2\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U)))} - \frac{\alpha+\beta}{\alpha-\beta}, \tag{13}$$

where $\widehat{\mathbb{C}}$ is the extended plane.

We now define a family of mappings, $\mathbb{G}$, which bears some resemblance to the convex mappings in the plane. The question is, how much? The lemmas which follow the definition lead up to two theorems which assert that $\mathbb{G}$ is between the convex mappings and the starlike mappings.

Definition 3. Let

$$\mathbb{G} = \{f \in \mathbb{S}_n : \mathrm{Re}\{G_f(\alpha, \beta)\} > 0, \text{ for all } \alpha, \beta \in \Delta \text{ and any } U \in S^{2n-1}\}.$$

We call this family of mappings the "Quasi-Convex Mappings, Type A".

Lemma 3. *The mapping $G_f(\alpha,\beta)$ is analytic in α and β.*

Proof. It suffices to show that there is a removable singularity at $\alpha=\beta$. We expand $f(\beta U)$ about αU to obtain

$$f(\beta U)=f(\alpha U)+Df(\alpha U)((\beta-\alpha)U)+\frac{1}{2}D^2f(\alpha U)([(\beta-\alpha)U]^2)+o((\beta-\alpha)^2).$$

Therefore

$$\begin{aligned} &Df(\alpha U)^{-1}(f(\alpha U)-f(\beta U))\\ &\quad= -Df(\alpha U)^{-1}(Df(\alpha U)((\beta-\alpha)U)+\frac{1}{2}D^2f(\alpha U)([(\beta-\alpha)U]^2)\\ &\qquad\qquad + o((\beta-\alpha)^2)))\\ &\quad= (\alpha-\beta)Df(\alpha U)^{-1}(Df(\alpha U)(U)+\frac{1}{2}(\beta-\alpha)D^2f(\alpha U)(U,U)\\ &\qquad\qquad + o((\beta-\alpha)))\\ &\quad= (\alpha-\beta)(U+\frac{1}{2}(\beta-\alpha)Df(\alpha U)^{-1}(D^2f(\alpha U)(U,U)+o(\beta-\alpha)). \end{aligned}$$

This gives

$$\begin{aligned} &G_f(\alpha,\beta)\\ &\quad= \frac{2\alpha-(\alpha+\beta)\ell_U[U+\frac{1}{2}(\beta-\alpha)Df(\alpha U)^{-1}(D^2f(\alpha U)(U,U))+o(\beta-\alpha)]}{(\alpha-\beta)\ell_U(U+\frac{1}{2}(\beta-\alpha)Df(\alpha U)^{-1}(D^2f(\alpha U)(U,U))+o(\beta-\alpha))}\\ &\quad= \frac{2\alpha-(\alpha+\beta)(1+\frac{1}{2}(\beta-\alpha)\ell_U[Df(\alpha U)^{-1}(D^2f(\alpha U)(U,U))+o(\beta-\alpha)])}{(\alpha-\beta)\ell_U(U+\frac{1}{2}(\beta-\alpha)Df(\alpha U)^{-1}(D^2f(\alpha U)(U,U))+o(\beta-\alpha))}\\ &\quad= \frac{(\alpha-\beta)(1+\frac{1}{2}(\beta+\alpha)\ell_U[Df(\alpha U)^{-1}(D^2f(\alpha U)(U,U))+o(\beta-\alpha)])}{(\alpha-\beta)\ell_U(U+\frac{1}{2}(\beta-\alpha)Df(\alpha U)^{-1}(D^2f(\alpha U)(U,U))+o(\beta-\alpha))}. \end{aligned}$$

Taking limits,

$$\begin{aligned} &\lim_{\beta\to\alpha} G_f(\alpha,\beta) \qquad (14)\\ &\quad= 1+\alpha\ell_U Df(\alpha U)^{-1}(D^2f(\alpha U)(U,U))\\ &\quad= \frac{1}{\alpha}\ell_U(\alpha U+Df(\alpha U)^{-1}(D^2f(\alpha U)(\alpha U,\alpha U)))\\ &\quad= \frac{1}{|\alpha|}\ell_{\alpha U}(\alpha U+Df(\alpha U)^{-1}(D^2f(\alpha U)(\alpha U,\alpha U))) \text{ where } \ell_{\alpha U}\in T(\alpha U)\\ &\quad= \frac{1}{|\alpha|}\ell_{\alpha U}(Df(\alpha U)^{-1}(D^2f(\alpha U)(\alpha U,\alpha U)+Df(\alpha U)(\alpha U))), \qquad (15) \end{aligned}$$

which is well defined. We conclude from (14) that G_f is indeed analytic in α and β. □

The next theorem asserts a result which was really the motivation for considering the family $\mathbb{G}$.

Theorem 3.1. *Let $f\in\mathbb{S}_n$, and assume f is convex. Then $f\in\mathbb{G}$.*

Proof. Given $f\in\mathbb{S}_n$, from Theorem 1.3 we have that if f is convex, then $\operatorname{Re}\{\ell_Z(Df(Z)^{-1}(f(Z)-f(V)))\}>0$ where $\|V\|<\|Z\|<1$ and $\ell_Z\in T(Z)$.

By considering the one-dimensional cross-section of B, C_U, we have

$$\begin{aligned} f \text{ convex} \quad &\Rightarrow \quad \mathrm{Re}\left\{\ell_{\alpha U}(Df(\alpha U)^{-1}(f(\alpha U)-f(\beta U)))\right\} > 0 \text{ where } |\beta|<|\alpha| \\ &\Rightarrow \quad \mathrm{Re}\left\{\frac{|\alpha|}{\alpha}\ell_U(Df(\alpha U)^{-1}(f(\alpha U)-f(\beta U)))\right\} > 0 \end{aligned}$$

since corresponding to each ℓ_U we have an $\ell_{\alpha U}(\cdot) = \frac{|\alpha|}{\alpha}\ell_U(\cdot)$ in $T(\alpha U)$. Thus

$$\begin{aligned} f \text{ convex} \quad &\Rightarrow \quad \mathrm{Re}\left\{\frac{1}{\alpha}\ell_U(Df(\alpha U)^{-1}(f(\alpha U)-f(\beta U)))\right\} > 0 \\ &\Rightarrow \quad \mathrm{Re}\left\{\frac{2\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U)-f(\beta U)))}\right\} > 0 \text{ for } |\beta|<|\alpha|. \end{aligned}$$

As before, if we let $|\beta| = |\alpha|$ with $\beta \neq \alpha$ we have

$$\mathrm{Re}\left\{\frac{2\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U)-f(\beta U)))}\right\} \geq 0.$$

If $|\alpha| = |\beta| = r$, then $\alpha = re^{i\theta}$, $\beta = re^{i\phi}$ for some $\theta, \phi \in \mathbf{R}$ and

$$\frac{\alpha+\beta}{\alpha-\beta} = -i\frac{\cos\frac{1}{2}(\theta-\phi)}{\sin\frac{1}{2}(\theta-\phi)}.$$

Hence $\mathrm{Re}\left\{\dfrac{\alpha+\beta}{\alpha-\beta}\right\} = 0$. Thus for $|\alpha| = |\beta|$, $\alpha \neq \beta$ we have

$$\begin{aligned} &\mathrm{Re}\left\{\frac{2\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U)-f(\beta U)))}\right\} \geq 0 \\ &\Leftrightarrow \quad \mathrm{Re}\left\{\frac{2\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U)-f(\beta U)))} - \frac{\alpha+\beta}{\alpha-\beta}\right\} \geq 0. \end{aligned}$$

That is $\mathrm{Re}\{G_f(\alpha,\beta)\} \geq 0$ for $|\alpha| = |\beta|$, $\alpha \neq \beta$. As we have seen in Lemma 3 G_f is analytic in both α and β. It follows that $\mathrm{Re}\{G_f(\alpha,\beta)\}$ is harmonic on $\Delta \times \Delta$.

Keeping α fixed and varying β, we apply the minimum principle for harmonic functions to assert that $\mathrm{Re}\{G_f(\alpha,\beta)\}$ cannot attain its minimum at an interior point, *i.e.* when $|\beta| < |\alpha|$. Similarly, holding β fixed and varying α, with $|\alpha| < |\beta|$, we obtain the same result for α. We conclude that on the whole polydisk $\Delta \times \Delta$, $\mathrm{Re}\{G_f(\alpha,\beta)\} > 0$. Hence $f \in \mathbb{G}$. □

Theorem 3.2. *If $f \in \mathbb{G}$, then f is starlike.*

Proof. If $f \in \mathbb{G}$, then $\mathrm{Re}\{G_f(\alpha,\beta)\} > 0$ for all $\alpha, \beta \in \Delta$. Consider the case when $\beta = 0$. Then

$$\begin{aligned} \mathrm{Re}\{G_f(\alpha,0)\} \quad &= \quad \mathrm{Re}\left\{\frac{2\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U)))} - 1\right\} > 0 \\ &\Rightarrow \quad \mathrm{Re}\left\{\frac{\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U)))}\right\} > \frac{1}{2} \\ &\Rightarrow \quad \mathrm{Re}\left\{\frac{1}{\alpha}\ell_U(Df(\alpha U)^{-1}(f(\alpha U)))\right\} > 0 \\ &\Rightarrow \quad \mathrm{Re}\,\frac{1}{|\alpha|}\{\ell_{\alpha U}(Df(\alpha U)^{-1}(f(\alpha U)))\} > 0 \end{aligned}$$

since there is a 1-1 correspondence between $T(\alpha U)$ and $T(U)$ given by $\ell_{\alpha U}(\cdot) = \frac{|\alpha|}{\alpha}\ell_U(\cdot)$. Thus

$$\operatorname{Re} G_f(\alpha, 0) > 0 \Rightarrow \operatorname{Re}\left\{\ell_{\alpha U}(Df(\alpha U)^{-1}(f(\alpha U)))\right\} > 0$$

and this is the condition for starlikeness from (1.2). □

The condition (8) led us to our definition of the family $\mathbb{G}$. An obvious question is, why not use the more common characterization of K, namely (7)? The analogous condition to this is

$$\operatorname{Re}\left\{\ell_Z(Df(Z)^{-1}(D^2f(Z)(Z,Z) + Df(Z)))\right\} > 0.$$

This leads us to define a new family of mappings, F. Naturally we will want to examine the relationship between F and $\mathbb{G}$. F, as we will see later, is defined by a local condition, whereas $\mathbb{G}$ is defined by a global condition. In the plane they are one and the same, but what about higher dimensions?

Further motivation comes from the derivation of (7). The condition

$$\operatorname{Re}\{zf''(z)/f'(z) + 1\} > 0$$

for convexity in the plane is equivalent to saying that the curvature of $f(z)$ is always positive for $z = re^{it}$ with r fixed and t real. When we generalize this to the image of $C_U = \{\alpha U : \|U\| = 1, \alpha \in \Delta\}$, and use a 2-norm, we obtain an expression which is similar to that in the plane. That is, the condition which ensures that the curvature of $f(Ze^{it})$, where $Z \in C_U$ with Z fixed and for some U, is always positive leads us to the same condition.

Let $r(t) = f(Ze^{it})$. Then $r'(t) = iDf(Ze^{it})(Ze^{it})$ and

$$r''(t) = -(D^2f(Ze^{it})(Ze^{it}, Ze^{it}) + Df(Ze^{it})(Ze^{it})).$$

Since $r''(t) = a_T(t)\mathbf{T}(t) + a_N(t)\mathbf{N}(t)$ where $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are the unit tangential and unit normal (inward) components to the curve $r(t)$. Also $a_N(t) = \kappa\|r'(t)\|^2$ where κ is the curvature and $a_N(t) = \operatorname{Re}\langle r''(t), \mathbf{N}(t)\rangle$.

$$\mathbf{N}(t) = -\frac{(Df(Ze^{it})^{-1})^*(Ze^{it})}{\|(Df(Ze^{it})^{-1})^*(Ze^{it})\|},$$

where $(Df(Ze^{it})^{-1})^*$ is the adjoint of the derivative. Hence

$$\begin{aligned} &\kappa\|Df(Ze^{it})(Ze^{it})\|^2 \\ &\quad = \operatorname{Re}\left\langle D^2f(Ze^{it})(Ze^{it}, Ze^{it}) + Df(Ze^{it})(Ze^{it}), \frac{(Df(Ze^{it})^{-1})^*(Ze^{it})}{\|(Df(Ze^{it})^{-1})^*(Ze^{it})\|}\right\rangle, \\ \kappa &= \frac{\operatorname{Re}\langle Df(Ze^{it})^{-1}(D^2f(Ze^{it})(Ze^{it}, Ze^{it}) + Df(Ze^{it})(Ze^{it})), Ze^{it}\rangle}{\|Df(Ze^{it})(Ze^{it})\|^2\|(Df(Ze^{it})^{-1})^*(Ze^{it})\|} \end{aligned}$$

Hence for any curve $X(t) = Ze^{it}$,

$$\operatorname{Re}\langle Df(X)^{-1}(D^2f(X)(X,X) + Df(X)(X)), X\rangle > 0$$

if and only if the curvature of $f(X(t))$ is positive.

This leads us to the following definitions.

Definition 4. Let $F_f(Z) = \ell_Z(Df(Z)^{-1}(D^2f(Z)(Z,Z) + Df(Z)(Z)))$ where $\ell_Z \in T(Z)$.

Definition 5. Let $\mathbb{F} = \{f \in \mathbb{S}_n : \operatorname{Re}\{F_f(Z)\} > 0 \text{ for all } Z \in B\}$. We call this family of mappings the "Quasi-Convex Mappings, Type B".

The first relationship between $\mathbb{F}$ and $\mathbb{G}$ we prove is that $\mathbb{G}$ is a subset of $\mathbb{F}$.

Theorem 3.3. *If $f \in \mathbb{G}$, then $f \in \mathbb{F}$.*

Proof. This follows easily from (15) in Lemma 3. □

As we have seen, a mapping which has a convex function of one variable in each of its coordinates is not necessarily convex. We prove here that for any absolute norm such mappings are Quasi-Convex.

Theorem 3.4. *Let $f : B \subset \mathbb{C}^n \to \mathbb{C}^n$ be defined by $f(Z) = (f_1(z_1), \dots , f_n(z_n))$ where $Z = (z_1, \dots , z_n)$ and $f_j \in K$, for each $j = 1, 2, \dots , n$. Then $f \in \mathbb{G}$ in any absolute norm. (That is, any norm for which $|z_j| \le |w_j|$ for each j implies that $\|Z\| \le \|W\|$.)*

Proof. We know that $Df(Z) = \operatorname{diag}\left\{f_j'(z_j)\right\}_{j=1}^n$ and since $f_j'(z_j) \neq 0$ for all j, $Df(Z)$ is nonsingular. Thus $Df(Z)^{-1} = \operatorname{diag}\left\{\dfrac{1}{f_j'(z_j)}\right\}_{j=1}^n$.

Let $U \in S^{2n-1}$, and $\alpha, \beta \in \Delta$ and define $W : B \times B \to \mathbb{C}^n$ by

$$W(\alpha U, \beta U) = Df(\alpha U)^{-1}\left(\frac{f(\alpha U) - f(\beta U)}{\alpha}\right).$$

Then

$$W(\alpha U, \beta U) = \left(\frac{f_j(\alpha u_j) - f_j(\beta u_j)}{\alpha f_j'(\alpha U)}\right)_{j=1}^n, \text{ where } U = (u_1, \dots , u_n).$$

Let $|\alpha| = r < 1$ and let $\beta = \gamma\alpha$ with $|\gamma| < 1$. For $t \in [0, 1]$, let

$$\begin{aligned} F_j(\alpha u_j, t) &= (1-t)f_j(\alpha u_j) + tf_j(\beta u_j) \\ &= (1-t)f_j(\alpha u_j) + tf_j(\gamma\alpha u_j). \end{aligned}$$

By the convexity of each of the f_j, F_j is subordinate to f_j on Δ for each $t \in [0, 1]$. Hence $F(\alpha U, t)$ is subordinate to $f(\alpha U)$ for $\alpha U \in B$ and $t \in [0, 1]$. (The norm we have chosen guarantees this). We have,

$$F(\alpha U, 0) = (F_j(\alpha u_j, 0))_j = (f_j(\alpha u_j))_j = f(\alpha U).$$

We now take the following limits,

$$\begin{aligned} &\lim_{t\to 0^+}\left(\frac{F(\alpha U, 0) - F(\alpha U, t)}{t}\right) \\ &= \lim_{t\to 0^+}\left(\frac{f_j(\alpha u_j) - (1-t)f_j(\alpha u_j) - tf_j(\gamma\alpha u_j)}{t}\right)_j \\ &= (f_j(\alpha u_j) - f_j(\gamma\alpha u_j))_j \\ &= G(\alpha U), \text{ say, which is holomorphic.} \end{aligned}$$

Hence by Lemma 2, $G(\alpha U) = Df(\alpha U)(V(\alpha U))$ with $V \in N_0$

$$\begin{aligned} V(\alpha U) &= Df(\alpha U)^{-1}(G(\alpha U)) \\ &= \left(\frac{f_j(\alpha u_j) - f_j(\gamma \alpha u_j)}{f_j'(\alpha U)} \right)_j \\ &= \left(\frac{f_j(\alpha u_j) - f_j(\beta u_j)}{f_j'(\alpha U)} \right)_j \\ &= \alpha W(\alpha U, \beta U). \end{aligned}$$

Hence $\alpha W(\alpha U, \beta U) \in N_0$ which means that

$$\operatorname{Re}\{\ell_{\alpha U}(\alpha W(\alpha U, \beta U))\} > 0,$$

where $\ell_{\alpha U} \in T(U)$.

Since for each $\ell_{\alpha U} \in T(\alpha U)$ there is a corresponding $\ell_U \in T(U)$ related by $\ell_{\alpha U}(\cdot) = \frac{|\alpha|}{\alpha}(\cdot)$, we have $\operatorname{Re}\{\ell_U(W(\alpha U, \beta U))\} > 0$. Thus

$$\operatorname{Re}\left\{ \frac{2\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U)))} \right\} > 0,$$

and it follows by a similar argument as in (3.1) that

$$\operatorname{Re}\left\{ \frac{2\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U) - f(\alpha U)))} - \frac{\alpha + \beta}{\alpha - \beta} \right\} > 0.$$

Hence $f \in \mathbb{G}$. □

Theorem 3.5. *Let B be the unit ball in $\mathbb{C}^n$ with a p-norm with $1 \le p \le \infty$. Let F be a mapping $F : B \to \mathbb{C}^n$ with one of its coordinate maps, f_k, a function of one variable only. It is a necessary condition for $F \in \mathbb{F}$ that $f_k \in K$.*

Proof. Without loss of generality we can assume that

$$F(Z) = (f(z), f_2(Z), \ldots, f_n(Z)), \text{ where } Z = (z, z_2, \ldots, z_n).$$

$$DF(Z) = \begin{bmatrix} f'(z) & 0_{n-1} \\ A_{n-1} & B_{(n-1)\times(n-1)} \end{bmatrix},$$

$$DF(Z)^{-1} = \begin{bmatrix} 1/f'(z) & 0_{n-1} \\ C_{n-1} & D_{(n-1)\times(n-1)} \end{bmatrix},$$

$$D^2F(Z)(Z, Z) = \begin{bmatrix} z^2 f''(z) \\ E_{n-1} \end{bmatrix}.$$

Choose $Z = (z, 0, \ldots, 0) = (re^{i\theta}, 0, \ldots, 0)$. Then the functional $\ell_Z(U) = e^{-i\theta}u_1$ is in $T(Z)$ and

$$\begin{aligned} &\operatorname{Re}\ell_Z(DF(Z)^{-1}(D^2F(Z)(Z, Z) + DF(Z)(Z))) \\ &= \operatorname{Re}\left\{ e^{-i\theta} \begin{bmatrix} re^{i\theta} \\ 0_{n-1} \end{bmatrix} + e^{-i\theta}[1/f'(z)\ 0_{n-1}] \begin{bmatrix} z^2 f''(z) \\ H_{n-1} \end{bmatrix} \right\} \\ &= \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \text{ if and only if } f \text{ is convex.} \end{aligned}$$

□

Corollary 1. *If $f : B \to \mathbb{C}^n$ where $B \in \ell^p(n)$, $1 \le p \le \infty$ is of the form $f(Z) = (f_1(z_1), \dots, f_n(z_n))$ where for each $j = 1, \dots, n$, $f_j \in S$, then $f \in \mathbb{G}$ (and $\mathbb{F}$) if and only if $f_j \in K$ for each j.*

Proof. The result follows immediately from Theorem 3.4 and Theorem 3.5. □

Theorem 3.6. *Let $f : B \to \mathbb{C}$ with $B \subset \mathbb{C}^n$ be holomorphic. Define $F : B \to \mathbb{C}^n$ by $F(Z) = f(Z)Z$. Further, given $U \in S^{2n-1}$, define $g_U : \Delta \to \mathbb{C}$ by $g_U(\alpha) = \alpha f(\alpha U)$. Then:*

1. *$F \in G$ if and only if $g_U \in K$ for each $U \in S^{2n-1}$.*
2. *$F \in F$ if and only if $g_U \in K$ for each $U \in S^{2n-1}$.*

Proof. Since $F(Z) = f(Z)(Z)$ where $Z^T = (z_1 \dots, z_n)$ we have

$$\begin{aligned} DF(Z) &= Z\nabla f(Z)^T + f(Z)I \\ &= Z\nabla f^T + fI \text{ (for simplicity).} \end{aligned} \tag{16}$$

It is easy to check that

$$DF(Z)^{-1} = \frac{1}{f(f + \nabla f^T Z)}[(f + \nabla f^T Z)I - Z\nabla f^T] \tag{17}$$

where $\nabla f = \left(\dfrac{\partial f}{\partial z_1}, \dots, \dfrac{\partial f}{\partial z_n}\right)^T$.

1. From $F(Z) = f(Z)Z$, we write

$$F(\alpha U) - F(\beta U) = (g_U(\alpha) - g_U(\beta))U.$$

Also, since

$$g_U'(\alpha) = f(\alpha U) + \nabla f(\alpha U)^T(\alpha U), \tag{18}$$

we have

$$\begin{aligned} DF(\alpha U)^{-1} &= \frac{(f(\alpha U) + \nabla f(\alpha U)^T(\alpha U))I - \alpha U\nabla f(\alpha U)^T}{f(\alpha U)(f(\alpha U) + \nabla f(\alpha U)^T(\alpha U))} \\ &= \frac{g_U'(\alpha)I - \alpha U\nabla f(\alpha U)^T}{f(\alpha U)(g_U'(\alpha))}. \end{aligned}$$

So

$$\begin{aligned} &DF(\alpha U)^{-1}(F(\alpha U) - F(\beta U)) \\ &= \frac{g_U'(\alpha)(g_U(\alpha) - g_U(\beta))U - (\alpha U\nabla f(\alpha U)^T)(g_U(\alpha) - g_U(\beta))U}{f(\alpha U)g_U'(\alpha)} \\ &= \frac{g_U(\alpha) - g_U(\beta)}{g_U'(\alpha)}U. \end{aligned}$$

It follows that

$$\ell_U(DF(\alpha U)^{-1}(F(\alpha U) - F(\beta U))) = \frac{g_U(\alpha) - g_U(\beta)}{g_U'(\alpha)} \tag{19}$$

for $\ell_U \in T(U)$.

Therefore

$$G_F(\alpha, \beta) = \frac{2\alpha g_U'(\alpha)}{g_U(\alpha) - g_U(\beta)} - \frac{\alpha + \beta}{\alpha - \beta},$$

and so $\operatorname{Re}\{G_F(\alpha,\beta)\} > 0$ if and only if $g_U \in K$ by (7).

2. Let $\widehat{f}(Z) = f(Z) + \nabla f(Z)^T Z$. Then

$$DF(Z)(Z) = Z\nabla f(Z)^T Z + f(Z)Z = \widehat{f}(Z)Z.$$

Differentiating again,

$$\begin{aligned} D^2F(Z)(Z,\cdot) + DF(Z)(\cdot) &= (Z\nabla\widehat{f}(Z)^T + \widehat{f}(Z)I)(\cdot) \\ D^2F(Z)(Z,Z) + DF(Z)(Z) &= (\nabla\widehat{f}^T Z + \widehat{f})Z. \end{aligned}$$

So we have

$$\begin{aligned} & DF(Z)^{-1}(D^2F(Z)(Z,Z) + DF(Z)(Z)) \\ &= \frac{1}{f\widehat{f}}(\widehat{f}I - Z\nabla f^T)(\nabla\widehat{f}^T Z + \widehat{f})Z \\ &= \frac{(\nabla\widehat{f}^T Z + \widehat{f})}{f\widehat{f}}(\widehat{f} - \nabla f^T Z)Z \\ &= \frac{(\nabla\widehat{f}^T Z + \widehat{f})}{\widehat{f}}Z. \end{aligned}$$

For $\ell_Z \in T(Z)$ we obtain

$$\ell_Z(DF(Z)^{-1}(D^2F(Z)(Z,Z) + DF(Z)(Z))) \tag{20}$$

$$= \|Z\|\left(\frac{\nabla\widehat{f}^T Z}{\widehat{f}} + 1\right). \tag{21}$$

Given $U \in S^{2n-1}$ let $g_U(\alpha) = \alpha f(\alpha U)$. Then

$$\begin{aligned} g_U'(\alpha) &= f(\alpha U) + \alpha\nabla f(\alpha U)^T U \\ &= \widehat{f}(\alpha U), \\ \alpha g_U''(\alpha U) &= \nabla\widehat{f}(\alpha U)^T \alpha U. \end{aligned}$$

Therefore

$$\frac{\alpha g_U''(\alpha)}{g_U'(\alpha)} + 1 = \frac{\nabla\widehat{f}(\alpha U)^T(\alpha U)}{\widehat{f}(\alpha U)} + 1.$$

So from (21) with $Z = \alpha U$ we see that

$$\begin{aligned} & \ell_Z(DF(Z)^{-1}(D^2F(Z)(Z,Z) + DF(Z)(Z))) \\ &= |\alpha|\left(\frac{\alpha g_U''(\alpha)}{g_U'(\alpha)} + 1\right) \end{aligned}$$

We conclude that $F \in \mathbb{F}$ if and only if $g_U \in K$ for each $U \in S^{2n-1}$. □

The following corollary involves an interesting mapping. Let us define the mapping $F : B \to \mathbb{C}^n$ by $F(Z) = \dfrac{f(\ell(Z))}{\ell(Z)}Z$, where $f \in S$ and $\ell \in T(U)$, for some $U \in \mathbb{C}^n$ with $\|U\| = 1$. This mapping has the property that in the one-dimensional space $\{\alpha U : \alpha \in \Delta\}$ it is identical to the mapping in the plane. In any other one-dimensional space described by $\{\alpha V : \alpha \in \Delta,\ \ell(V) = 0\}$ we have $F(Z) = Z$. That is, the identity mapping. This follows from $f(z)/z$ having a removable singularity at $z = 0$. This is easily seen by the following computation.

$$F(\alpha U + \beta V) = \frac{f(\alpha)}{\alpha}(\alpha U + \beta V). \tag{22}$$

Hence if $\beta = 0$, $F(\alpha U) = f(\alpha)U$ and if $\alpha = 0$, $F(\beta V) = \beta V$.

Corollary 2. *Let $\ell \in T(U')$ for some $U' \in S_{2n-1}$. Define $f : B \to \mathbb{C}$ by*

$$f(Z) = \frac{h(\ell(Z))}{\ell(Z)}$$

where $h \in S$. Then $F : B \to \mathbb{C}^n$ given by $F(Z) = f(Z)Z$ is in $\mathbb{G}$ (or $\mathbb{F}$) if and only if $h \in K$.

Proof. Given $U \in S_{2n-1}$, we have

$$g_U(\alpha) = \alpha f(\alpha U) = \frac{h(\alpha\ell(U))}{\ell(U)}.$$

Then

$$\mathrm{Re}\left\{\alpha\frac{g_U''(\alpha)}{g_U'(\alpha)} + 1\right\} = \mathrm{Re}\left\{\alpha\ell(U))\frac{h''(\alpha\ell(U))}{h'(\alpha\ell(U))} + 1\right\} > 0 \text{ if and only if } h \in K.$$

The result follows from Theorem 3.6. □

Example 9. The function $f(Z) = (z + aw^2, w)$ where $Z = (z, w)$, $\|Z\|^p = |z|^p + |w|^p < 1$, $z, w \in \mathbf{C}$ is in $\mathbb{G}$ if and only if

$$|a| \le \frac{1}{2}\left(\frac{p^2-1}{4}\right)^{1/p}\left(\frac{p+1}{p-1}\right).$$

As before,

$$Df(Z) = \begin{bmatrix} 1 & 2aw \\ 0 & 1 \end{bmatrix},\ Df(Z)^{-1} = \begin{bmatrix} 1 & -2aw \\ 0 & 1 \end{bmatrix}.$$

Using $f \in G$ if and only if $\mathrm{Re}\, G_f(\alpha, \beta) > 0$ where

$$G_f(\alpha, \beta) = \frac{2\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U)))} - \frac{\alpha+\beta}{\alpha-\beta}$$

and $U = (z, w)$, $\|U\| = 1$, $|\alpha| < 1$, $|\beta| < 1$, $\alpha, \beta \in \mathbf{C}$. It follows that

$$\begin{aligned} &Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U)) \\ &= \begin{bmatrix} 1 & -2a\alpha w \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha z + a\alpha^2 w^2 - \beta z - a\beta^2 w^2 \\ \alpha w - \beta w \end{bmatrix} \\ &= (\alpha - \beta)\begin{bmatrix} z - a(\alpha-\beta)w^2 \\ w \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} &\ell_U(Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U))) \\ &= (\alpha - \beta)(\|U\| - a(\alpha-\beta)\ell_U((w^2, 0))) \\ &= (\alpha - \beta)(1 - a(\alpha-\beta)\ell_U((w^2, 0))). \end{aligned}$$

So

$$\begin{aligned} G_f(\alpha,\beta) &= \frac{2\alpha}{(\alpha-\beta)(1-a(\alpha-\beta)\ell_U((w^2,0)))} - \frac{\alpha+\beta}{\alpha-\beta} \\ &= \frac{2\alpha-(\alpha+\beta)(1-a(\alpha-\beta)\ell_U((w^2,0)))}{(\alpha-\beta)(1-a(\alpha-\beta)\ell_U((w^2,0)))} \\ &= \frac{1+a(\alpha+\beta)w^2\bar{z}}{1-a(\alpha-\beta)\ell_U((w^2,0))} \\ &= \left(1+\frac{a\ell_U((w^2,0))}{1+a\beta\ell_U((w^2,0))}\alpha\right)\Big/\left(1-\frac{a\ell_U((w^2,0))}{1+a\beta\ell_U((w^2,0))}\alpha\right) \\ &= \frac{1+b\alpha}{1-b\alpha}, \text{ where } b=\frac{a\ell_U((w^2,0))}{1+a\beta\ell_U((w^2,0))}. \end{aligned}$$

Since $\dfrac{1+b\alpha}{1-b\alpha} = \dfrac{1-|b\alpha|^2}{|1-b\alpha|^2} + \dfrac{2i\,\mathrm{Im}\,\{b\alpha\}}{|1-b\alpha|^2}$ it follows that $\mathrm{Re}\,\{G_f(\alpha,\beta)\geq 0\}$ if and only if $|b\alpha|\leq 1$. Thus we need

$$|b| = \left|\frac{a\ell_U((w^2,0))}{1+a\beta\ell_U((w^2,0))}\right| \leq 1.$$

Hence

$$|a||\ell_U((w^2,0))| \leq |1+a\beta\ell_U((w^2,0))|,$$

and in the worst case

$$|a||\ell_U((w^2,0))| \leq 1-|a||\ell_U((w^2,0))|.$$

That is, $2|a||\ell_U((w^2,0))|\leq 1$.

If we are using a p-norm, $1<p<\infty$, $\ell_U((x_1,x_2)) = |z|^{p-2}\bar{z}x_1 + |w|^{p-2}\bar{w}x_2$. Then $\ell_U((w^2,0)) = |z|^{p-2}\bar{z}w^2$ and $|\ell_U((w^2,0))| = |z|^{p-1}(1-|z|^p)^{2/p}$. Hence $2|a||\ell_U((w^2,0))|\leq 1$ if and only if

$$|a| \leq \frac{1}{2}\left(\frac{p^2-1}{4}\right)^{1/p}\left(\frac{p+1}{p-1}\right).$$

We note that if f is in $\mathbb{G}$, then f is starlike. □

Example 10. The function $f(Z) = (z+aw^2, w)$ where $Z=(z,w)$, $\|Z\|^p = |z|^p + |w|^p < 1$, $z,w\in\mathbf{C}$ is in $\mathbb{F}$ if and only if

$$|a| \leq \frac{1}{2}\left(\frac{p^2-1}{4}\right)^{1/p}\left(\frac{p+1}{p-1}\right).$$

We have that $f\in\mathbb{F}$ if and only if

$$\mathrm{Re}\,\ell_Z(Df(Z)^{-1}(D^2f(Z)(Z,Z)+Df(Z)(Z))) > 0.$$

Therefore,

$$\begin{aligned} &Df(Z)^{-1}(D^2f(Z)(Z,Z)+Df(Z)(Z)) \\ &= \begin{bmatrix} 1 & -2aw \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2aw^2 \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix} \\ &= \begin{bmatrix} z+2aw^2 \\ w \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned}&\operatorname{Re}\left\{\ell_Z(Df(Z)^{-1}(D^2f(Z)(Z,Z)+Df(Z)(Z)))\right\}\\&= \operatorname{Re}\{\|Z\|+2a\ell_Z((w^2,0))\}\\&= \operatorname{Re}\{\|Z\|+2a\ell_Z((w^2,0))\}\\&\geq \operatorname{Re}\{1+2a\ell_Z((w^2,0))\} \text{ (minimum principle)}\\&\geq \operatorname{Re}\{1-2|a||\ell_Z((w^2,0))|\}.\end{aligned}$$

This is the same condition as for the family $\mathbb{G}$ and so the same bound applies. □

It should be noted that, as with $\mathbb{G}$, $f(z,w)=(z+aw^2,w)\in\mathbb{F}\Rightarrow f$ is starlike.

Example 11. The mapping $f(z,w)=(z+azw,w)$ with $(z,w)\in B\subset\mathbb{C}^2$ with a p-norm is in $\mathbb{G}$ if and only if

$$|a|\leq\left(\frac{2}{3}(p+1)\right)^{1/p}\left(\frac{p+1}{3p}\right).$$

We know that $f\in\mathbb{G}$ if and only if $\operatorname{Re}G_f(\alpha,\beta)>0$.

$$Df(\alpha U)^{-1}(f(\alpha U)-f(\beta U))=(\alpha-\beta)\begin{bmatrix} z-\frac{(\alpha-\beta)awz}{1+aw\alpha}\\ w\end{bmatrix}$$

where $U=(z,w),\ \|U\|=1$.

If we use a p-norm,

$$\ell_U\left((\alpha-\beta)\left(z-\frac{(\alpha-\beta)awz}{1+aw\alpha},w\right)\right)=(\alpha-\beta)\left(1-(\alpha-\beta)\frac{aw|z|^p}{1+aw\alpha}\right).$$

Hence

$$\begin{aligned}G_f(\alpha,\beta) &= \frac{2\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U)-f(\beta U))}-\frac{\alpha+\beta}{\alpha-\beta}\\ &= \frac{2\alpha}{\frac{(\alpha-\beta)}{1+aw\alpha}(1+aw\alpha-(\alpha-\beta)aw|z|^p)}-\frac{\alpha+\beta}{\alpha-\beta}\\ &= \frac{1+aw\alpha+(\alpha+\beta)aw|z|^p}{1+aw\alpha-(\alpha-\beta)aw|z|^p}\\ &= \frac{1+\beta aw|z|^p+aw(1+|z|^p)\alpha}{1+\beta aw|z|^p-aw(1-|z|^p)\alpha}\\ &= \left(1+\frac{aw\alpha|z|^p}{1+\beta aw|z|^p+aw\alpha}\right)\Big/\left(1-\frac{aw\alpha|z|^p}{1+\beta aw|z|^p+aw\alpha}\right).\end{aligned}$$

The real part of this is positive if and only if

$$\left|\frac{aw\alpha|z|^p}{1+\beta aw|z|^p+aw\alpha}\right|\leq 1.$$

Hence we need $|a||w||\alpha||z|^p\leq|1+\beta aw|z|^p+aw\alpha|$. The worst case is when $\alpha aw=\beta aw=-|aw|$, and we have to find a such that $|a||w||z|^p\leq 1-|a||w||z|^p-|a||w|$. We obtain $|a||w|(3-2|w|^p)\leq 1$ from which we find that

$$|a|\leq\left(\frac{2}{3}(p+1)\right)^{1/p}\left(\frac{p+1}{3p}\right).$$

In particular we have the following values of a.

If $p = 1$ $|a| \leq 8/9$,
If $p = 2$ $|a| \leq 1/\sqrt{2}$,
If $p = \infty$ $|a| \leq 1/3$. □

Note that for $p = 2$ the values obtained for a are the same for $\mathbb{G}$ as for the convex mappings.

Example 12. The mapping $f(z, w) = (z + azw, w)$ is in $\mathbb{F}$ if and only if

$$|a| \leq \left(\frac{2}{3}(p+1)\right)^{1/p} \left(\frac{p+1}{3p}\right).$$

This is the same result as in Example 11. This follows directly from the observation that the worst case in that example occurs when $\alpha = \beta$. □

4. Some bounds on the quasi-convex mappings

We now turn our attention to finding information on the family $\mathbb{G}$ as a whole, Theorem 4.1 gives us some uniform bounds, in the Euclidean norm, on $\mathbb{G}$. We first prove two lemmas.

Lemma 4. *Let $f : B \to \mathbb{C}^n$ be holomorphic and univalent on B. Let $U \in \mathbb{C}^n$ with $\|U\| = 1$ and let $\alpha \in \Delta$. Then necessary conditions for $\|f(Z)\|$ to have a local maximum or minimum on $\{Z : \|Z\| = r < 1\}$ at $Z = \alpha U$, $|\alpha| = r$ are*

$$\operatorname{Im}\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle = 0, \tag{23}$$

and

$$\langle Df(\alpha U)(\alpha V), f(\alpha U)\rangle = 0, \tag{24}$$

where $V \in \mathbb{C}^n$, $\|V\| = 1$ and $\langle U, V\rangle = 0$.

Proof. Let $\alpha = re^{i\theta}$ where r is fixed and θ varies.

$$\begin{aligned}
\frac{d}{d\theta}\left(\|f(\alpha U)\|^2\right) &= \frac{d}{d\theta}\langle f(\alpha U), f(\alpha U)\rangle \\
&= \langle Df(\alpha U)(i\alpha U), f(\alpha U)\rangle + \langle f(\alpha U), Df(\alpha U)(i\alpha U)\rangle \\
&= 2\operatorname{Re}\langle Df(\alpha U)(i\alpha U), f(\alpha U)\rangle \\
&= 2\operatorname{Re}\{i\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle\} \\
&= -2\operatorname{Im}\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle.
\end{aligned}$$

When a maximum (or minimum) of $\|f(\alpha U)\|^2$ for $|\alpha| = r$ occurs,

$$\operatorname{Im}\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle = 0.$$

That is, $\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle$ is real.

Now fix α at a point where $\|f(\alpha U)\|$ has a maximum, $|\alpha| = r$, and vary Z by letting $Z(\theta) = \alpha(U\cos\theta + \lambda V \sin\theta)$, where $\langle U, V\rangle = 0$, $\|V\| = 1$, $|\lambda| = 1$.

$$\begin{aligned}
&\frac{d}{d\theta}\left(\langle f(Z(\theta)), f(Z(\theta))\rangle\right) \\
&= \langle Df(Z(\theta))(Z'(\theta)), f(Z(\theta))\rangle + \langle f(Z(\theta)), Df(Z(\theta))(Z'(\theta))\rangle \\
&= 2\operatorname{Re}\langle Df(Z(\theta))(Z'(\theta)), f(Z(\theta))\rangle.
\end{aligned}$$

We want this to have a maximum at $\theta = 0$. Hence we need

$$2\,\mathrm{Re}\langle Df(Z(\theta))(Z'(\theta)), f(Z(\theta))\rangle|_{\theta=0} = 0.$$

Since $Z'(0) = \alpha(U(-\sin\theta)+\lambda V\cos\theta)|_{\theta=0}$, $Z'(0) = \lambda\alpha V$, $Z(0) = \alpha U$, this becomes

$$2\,\mathrm{Re}\langle Df(\alpha U)(\lambda\alpha V), f(\alpha U)\rangle = 0.$$

Hence $\mathrm{Re}\,\{\lambda\langle Df(\alpha U)(\alpha V), f(\alpha U)\rangle\} = 0$, for all λ, $|\lambda| = 1$. Thus it follows that $|\langle Df(\alpha U)(\alpha V), f(\alpha U)\rangle| = 0$. Consequently,

$$\langle Df(\alpha U)(\alpha V), f(\alpha U)\rangle = 0.$$

□

Lemma 5. *Let $(r_n)_{n=1}^{\infty}$ be a monotone increasing sequence of positive numbers converging to 1. Let $f \in G$ and define $f_n(Z) = (1/r_n)f(r_nZ)$. Then*

1. *$f_n \in G$, and*
2. *$f_n \to f$ uniformly on compact subsets of B.*

Proof. We first note that $Df_n(Z) = Df(r_nZ)$. Hence

$$\begin{aligned} G_{f_n}(\alpha,\beta) &= \frac{2\alpha}{\langle Df_n(\alpha U)^{-1}(f_n(\alpha U) - f_n(\beta U)), U\rangle} - \frac{\alpha+\beta}{\alpha-\beta} \\ &= \frac{2r_n\alpha}{\langle Df(r_n\alpha U)^{-1}(f(r_n\alpha U) - f_n(r_n\beta U)), U\rangle} - \frac{r_n\alpha + r_n\beta}{r_n\alpha - r_n\beta} \\ &= G_f(r_n\alpha, r_n\beta) \end{aligned}$$

and $f_n \in G$.

That $f_n \to f$ uniformly on compact subsets of B follows by a standard argument.

□

Theorem 4.1. *Let $f \in G$, then for all $Z \in B$, using the 2-norm,*

$$\frac{\|Z\|}{1+\|Z\|} \le \|f(Z)\| \le \frac{\|Z\|}{1-\|Z\|}.$$

Proof. Since $\mathrm{Re}\,G_f(\alpha,\beta) > 0$ and $G_f(0,\beta) = 1$ we can write

$$G_f(\alpha,\beta) = \frac{1+\alpha\omega(\alpha,\beta)}{1-\alpha\omega(\alpha,\beta)},$$

where $\alpha\omega(\alpha,\beta)$ is a Schwarz function. That is, $\alpha\omega(\alpha,\beta)$ is analytic for $\alpha,\beta \in \Delta$ and $|\alpha\omega(\alpha,\beta)| \le |\alpha|$. Thus $|\omega(\alpha,\beta)| \le 1$. We have

$$\frac{2\alpha}{\langle Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U)), U\rangle} - \frac{\alpha+\beta}{\alpha-\beta} = \frac{1+\alpha\omega(\alpha,\beta)}{1-\alpha\omega(\alpha,\beta)}.$$

Hence

$$\begin{aligned} \frac{2\alpha}{\langle Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U)), U\rangle} &= \frac{1+\alpha\omega(\alpha,\beta)}{1-\alpha\omega(\alpha,\beta)} + \frac{\alpha+\beta}{\alpha-\beta} \\ &= \frac{2\alpha(1-\beta\omega(\alpha,\beta))}{(\alpha-\beta)(1-\alpha\omega(\alpha,\beta))}. \end{aligned}$$

Thus

$$\langle Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U)), U\rangle = (\alpha-\beta)\frac{1-\alpha\omega(\alpha,\beta)}{1-\beta\omega(\alpha,\beta)}$$

and

$$Df(\alpha U)^{-1}(f(\alpha U)-f(\beta U))=(\alpha-\beta)\frac{1-\alpha\omega(\alpha,\beta)}{1-\beta\omega(\alpha,\beta)}U+\sum_{j=2}^{n}d_j(\alpha,\beta)V_j$$

where $d_j(\alpha,\beta)$ is analytic in α and β, $\langle V_j,U\rangle=0$ and $\langle V_j,V_k\rangle=0$ for $j\neq k$. Thus

$$(f(\alpha U)-f(\beta U))=(\alpha-\beta)\frac{1-\alpha\omega(\alpha,\beta)}{1-\beta\omega(\alpha,\beta)}Df(\alpha U)(U)+\sum_{j=2}^{n}d_j(\alpha,\beta)Df(\alpha U)(V_j).$$

Further, dividing by $\alpha-\beta$ and letting $\beta\to\alpha$ we have

$$Df(\alpha U)(U)=Df(\alpha U)(U)+\sum_{j=2}^{n}\lim_{\beta\to\alpha}\frac{d_j(\alpha,\beta)}{\alpha-\beta}Df(\alpha U)(V_j).$$

Therefore

$$\lim_{\beta\to\alpha}\frac{d_j(\alpha,\beta)}{\alpha-\beta}=0 \text{ for } j=2,\dots,n.$$

From this we conclude that $d_j(\alpha,\beta)=(\alpha-\beta)^2c_j(\alpha,\beta)$ where $c_j(\alpha,\beta)$ is analytic in α and β. So we can write

$$f(\alpha U)-f(\beta U)=(\alpha-\beta)\frac{1-\alpha\omega(\alpha,\beta)}{1-\beta\omega(\alpha,\beta)}Df(\alpha U)(U)+\sum_{j=2}^{n}(\alpha-\beta)^2c_j(\alpha,\beta)Df(\alpha U)(V_j). \tag{25}$$

From this we get two useful representations of $f(Z)$.

When $\beta=0$,

$$f(\alpha U)=\alpha(1-\alpha\omega(\alpha,0))Df(\alpha U)(U)+\sum_{j=2}^{n}\alpha^2c_j(\alpha,0)Df(\alpha U)(V_j), \tag{26}$$

and when $\alpha=0$,

$$f(\beta U)=\frac{\beta}{1-\beta\omega(0,\beta)}U-\sum_{j=2}^{n}\beta^2c_j(0,\beta)V_j. \tag{27}$$

Since

$$\begin{aligned}\|f(\beta U)\|^2 &= \langle f(\beta U),f(\beta U)\rangle\\ &\geq \frac{|\beta|^2}{|1-\beta\omega(0,\beta)|^2}\\ &\geq \frac{|\beta|^2}{(1+|\beta|)^2},\end{aligned}$$

we have the lower bound. □

To obtain the upper bound requires a lot more work. We note that if $|\omega(\alpha,\beta)|=1$, it follows that $\omega(\alpha,\beta)=e^{i\theta}$ for all $\alpha,\beta\in\Delta$ and θ is a real constant. If $\beta=|\beta|e^{-i\theta}$ then

$$\|f(\beta U)\|^2=\frac{|\beta|^2}{(1-|\beta|)^2}+|\beta|^4\sum_{j=2}^{n}|c_j(0,\beta)|^2.$$

Clearly, if $c_j(0,|\beta|e^{-i\theta}) \neq 0$ for some j, then $\|f(\beta U)\| > \dfrac{|\beta|}{1-|\beta|}$. Our approach will implicitly show that this does not happen.

Let $(r_n)_{n=1}^{\infty}$ and f_n be as in Lemma 5. Then from the lemma we know that $f_n \in \mathbb{G}$ and $f_n \to f$ uniformly on compact sets. In addition we will show that for each n, the $\omega_n(\alpha,\beta)$ associated with f_n as in (25) has the property $|\omega_n(\alpha,\beta)| < 1$. Once this is established it will suffice to show that the bound holds for mappings with $|\omega(\alpha,\beta)| < 1$.

To see that for any f_n, $|\omega_n(\alpha,\beta)| < 1$ we use

$$G_{f_n}(\alpha,\beta) = G_f(r_n\alpha, r_n\beta)$$

from Lemma 5 and

$$G_f(\alpha,\beta) = \frac{1+\alpha\omega(\alpha,\beta)}{1-\alpha\omega(\alpha,\beta)}.$$

From this we have $|\omega_n(\alpha,\beta)| = |r_n\omega(r_n\alpha, r_n\beta)|$. Therefore $|\omega_n(\alpha,\beta)| < 1$ since $r_n < 1$ and $|\omega(\alpha,\beta)| \leq 1$.

Now let $f \in \mathbb{G}$ have the property $|\omega(\alpha,\beta)| < 1$, for all $\alpha, \beta \in \Delta$ and let

$$T = \left\{ r : \|f(Z)\| \leq \frac{\|Z\|}{1-\|Z\|} \text{ for } \|Z\| < r \right\}.$$

We will show that T is both open and closed and conclude that $T = [0,1]$ for every f with the property that $|\omega(\alpha,\beta)| < 1$. Note that although T depends on f, for the sake of simplicity, our notation will not explicitly reflect this.

$T \neq \phi$ since $0 \in T$ (vacuously).

Next we show that there exists $\varepsilon > 0$ such that $[0,\varepsilon] \in T$.

We have seen that

$$\|f(\beta U)\|^2 = \frac{|\beta|^2}{|1-\beta\omega(0,\beta)|^2} + |\beta|^4 \sum_{j=2}^{n} |c_j(0,\beta)|^2.$$

Let us assume that $|\beta| \leq \frac{1}{2}$ and let $M = \max_{|\beta|\leq 1/2} \sum_{j=2}^{n} |c_j(0,\beta)|^2$. Also let $|\omega(0,\beta)| \leq \rho < 1$ for $|\beta| \leq \frac{1}{2}$. Hence

$$\|f(\beta U)\|^2 \leq \frac{|\beta|^2}{(1-\rho|\beta|)^2} + |\beta|^4 M,$$

and we need $\|f(\beta U)\|^2 \leq \dfrac{|\beta|^2}{(1-|\beta|)^2}$. We will find the conditions on $|\beta|$ for $\dfrac{|\beta|^2}{(1-\rho|\beta|)^2} + |\beta|^4 M \leq \dfrac{|\beta|^2}{(1-|\beta|)^2}$ and the required inequality will follow.

Easily,

$$\frac{1}{(1-\rho|\beta|)^2} + |\beta|^2 M \leq \frac{1}{(1-|\beta|)^2}$$

if and only if

$$(1-|\beta|)^2 + |\beta|^2 M (1-|\beta|)^2 (1-\rho|\beta|)^2 \leq (1-\rho|\beta|)^2.$$

Thus, it is sufficient to obtain

$$(1-|\beta|)^2 + |\beta|^2 M \leq (1-\rho|\beta|)^2.$$

Hence we have

$$\begin{aligned} 1-2|\beta|+|\beta|^2+|\beta|^2M &\le 1-2\rho|\beta|+\rho^2|\beta|^2, \\ |\beta|(1-\rho^2+M) &\le 2(1-\rho), \\ |\beta| &\le \frac{2(1-\rho)}{1-\rho^2+M}. \end{aligned}$$

Since $\frac{2(1-\rho)}{1-\rho^2+M} > 0$ we can choose $\varepsilon \le 1/2$ such that $0 < \varepsilon < \frac{2(1-\rho)}{1-\rho^2+M}$. Hence $[0,\varepsilon] \subset T$.

T is closed if r_0 is a limit point of T and $r_0 \notin T$, then there is a neighborhood, N, of r_0 such that $\|f(Z)\| > \frac{\|Z\|}{1-\|Z\|}$ for some Z with $\|Z\| \in N$, $\|Z\| = r_1 < r_0$. Now choose r_2 such that $r_1 < r_2 < r_0$ such that $r_2 \in T$ (r_2 exists since r_0 is a limit point). Then by the definition of T, $\|f(Z)\| \le \frac{\|Z\|}{1-\|Z\|}$ for all Z such that $\|Z\| < r_2$. This is contradiction and so $r_0 \in T$. Hence T is closed.

To establish that T is (relatively) open we will show that if $\|f(Z)\| = \frac{\|Z\|}{1-\|Z\|}$ for some $Z = Z_0$, $\|Z_0\| = r_0 \in T$, then it would mean that $\|f(Z)\| > \frac{\|Z\|}{1-\|Z\|}$ for some Z, $\|Z\| = r < r_0$. This would contradict the definition of T. So on B_{r_0}, $\|f(Z)\| < \frac{\|Z\|}{1-\|Z\|}$. Hence there is a sufficiently small $\delta > 0$ such that $r+\delta \in T$. It would follow that T is open.

Suppose $\|f(\alpha U)\| = \frac{|\alpha|}{1-|\alpha|}$ for some α, $|\alpha| = r_0$. Then since $\|f(\gamma U)\| \le \frac{|\gamma|}{1-|\gamma|}$ for all $|\gamma| = r < r_0$ (because $r \in T$), and since this is on the interior of B_{r_0}, $\|f(\alpha U)\|$ must be the maximum value of $\|f(Z)\|$ on ∂B_{r_0}. From Lemma 4 we have that at this maximum point

$$\langle Df(\alpha U)(V), f(\alpha U)\rangle = 0 \text{ where } \langle U, V\rangle = 0,$$

and

$$\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle > 0$$

by a suitable choice of coordinates.

From (26) we have

$$f(\alpha U) = \alpha(1-\alpha\omega(\alpha,0))Df(\alpha U)(U) + \sum_{j=2}^{n} \alpha^2 c_j(\alpha,0)Df(\alpha U)(V_j).$$

It follows that

$$\|f(\alpha U)\|^2 = \langle f(\alpha U), f(\alpha U)\rangle = \alpha(1-\alpha\omega(\alpha,0))\langle Df(\alpha U)(U), f(\alpha U)\rangle.$$

At this point $(1-\alpha\omega(\alpha,0)) > 0$.

By a suitable relabeling we can assume α is positive. We have

$$\begin{aligned}\frac{d}{dt}(\|f(tU)\|^2)\bigg|_{t=\alpha} &= \frac{d}{d\alpha}\langle f(\alpha U), f(\alpha U)\rangle \\ &= 2\operatorname{Re}\langle Df(\alpha U)(U), f(\alpha U)\rangle \\ &= \frac{2\|f(\alpha U)\|^2}{\alpha(1-\alpha\omega(\alpha,0))}.\end{aligned}$$

Hence

$$\frac{\frac{d}{dt}(\|f(tU)\|^2)}{\|f(tU)\|^2}\bigg|_{t=\alpha} = \frac{2}{\alpha(1-\alpha\omega(\alpha,0))},$$

therefore

$$\frac{d}{dt}\log(\|f(tU)\|^2)\bigg|_{t=\alpha} = \frac{2}{\alpha(1-\alpha\omega(\alpha,0))}.$$

That is,

$$\begin{aligned}\frac{d}{dt}\log(\|f(tU)\|)\bigg|_{t=\alpha} &= \frac{1}{\alpha(1-\alpha\omega(\alpha,0))} \\ &< \frac{1}{\alpha(1-\alpha)} \quad \text{since} \quad |\omega(\alpha,0)|<1. \qquad (28)\end{aligned}$$

Therefore, for t in a sufficiently small neighborhood of α, inequality (28) holds. That is,

$$\frac{d}{dt}\log\|f(tU)\| < \frac{1}{t(1-t)}.$$

Choose $0<\xi<\alpha$ in this neighborhood and integrate along the radial path $Z(t) = tU,\ t\in[\xi,\alpha]$.

$$\begin{aligned}\int_\xi^\alpha \frac{d}{dt}\log\|f(tU)\|\,dt &< \int_\xi^\alpha \frac{1}{t(1-t)}\,dt = \left[\log\frac{t}{1-t}\right]_\xi^\alpha, \\ \log\frac{\|f(\alpha U)\|}{\|f(\xi U)\|} &< \log\left(\frac{\alpha}{1-\alpha}\cdot\frac{1-\xi}{\xi}\right), \\ \frac{\|f(\alpha U)\|}{\|f(\xi U)\|} &< \frac{\alpha}{1-\alpha}\cdot\frac{1-\xi}{\xi}.\end{aligned}$$

But $\|f(\alpha U)\| = \dfrac{\alpha}{1-\alpha}$, and so $\|f(\xi U)\| > \dfrac{\xi}{1-\xi}$. This contradiction shows T is open, and we conclude that $T=[0,1]$.

Thus for any $f\in\mathbb{G}$ with $|\omega(\alpha,\beta)|<1$ the bound holds on B.

The theorem is now proved. □

These bounds are sharp as the following examples will show.

Example 13. The mapping $f: B\subset\mathbb{C}^2\to\mathbb{C}^2$ given by

$$f(z,w) = \left(\frac{z}{1-z}, \frac{w}{1-z}\right)$$

attains the bounds.

The function is convex and hence in $\mathbb{G}$. We note that

$$f(z,0)=\left(\frac{z}{1-z},0\right).$$

Since the mapping $\dfrac{z}{1-z}$ attains these bounds in the plane, then f will attain the bounds given by the theorem. □

The next example shows that several mappings attain this bound.

Example 14. Any mapping of the form $\left(\dfrac{z}{1-z},g(w)\right)$, where $g\in K$ is in $\mathbb{G}$ and attains the bounds.

Since the family $\mathbb{G}$ is locally uniformly bounded, it is normal and compact. Some of the implications of this are:

1. In the Taylor expansion of f, $f(Z)=Z+\sum_{k=2}^{\infty}P_k(Z)$, the $P_k(Z)$'s, which are homogenous polynomials in $z_1,\dots,z_n$, are uniformly bounded.
2. There are uniform bounds on the volume of the image of the ball of radius R, $R<1$.
3. There are uniform bounds on the determinant of the Jacobian of f.

It is known that for convex maps, $\|P_k\|\le 1$ for each k and the upper bound $\dfrac{\|Z\|}{1-\|Z\|}$ can be readily determined ([1]). However, in our families the mapping $(z+aw^2,w)$ can have $|a|=3\sqrt{3}/4\approx 1.3$ and so the bound $\|P_k\|\le 1$ does not hold for $\mathbb{F}$ or $\mathbb{G}$.

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