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Meet and Join in the Weak Order Lattice

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CHAPTER 9

ON INFS AND SUPS IN THE WEAK ORDER LATTICE

Finally, we focus on the proof of Theorem 1.4.2. Before we prove what was stated there, we have a good deal in the way of preliminaries to take care of. The discussion below is inspired almost exclusively by material contained in the work [3].

9.1 A Connection with Complete, Directed, Acyclic Graphs

Given $\omega \in S_n$, recall the set of non-inversions of ω ,

$$E(\omega) := \{(i, j) : i < j, \omega^{-1}(i) < \omega^{-1}(j)\},$$

and the set of inversions of ω ,

$$E^*(\omega) := \{(i, j) : i > j, \omega^{-1}(i) < \omega^{-1}(j)\}.$$

Note that ω is uniquely determined by its $E(\omega)$ (equivalently, by its $E^*(\omega)$). We have seen that, given permutations $\pi, \sigma \in S_n$, we have $\pi \leq \sigma$ in the *weak order* (written $\pi \preceq \sigma$) if and only if $E(\pi) \supseteq E(\sigma)$ (equivalently $E^*(\pi) \subseteq E^*(\sigma)$). It is beneficial to consider the sets $E(\omega)$ and $E^*(\omega)$ as directed edges in a complete, simple, labelled digraph. Namely, we define

$$G(\omega) = ([n], E(\omega) \sqcup E^*(\omega))$$

by joining i and j with an arc directed from i to j if $(i, j) \in E(\omega)$ ($(i, j) \in E^*(\omega)$ resp.). Note that $G(\omega)$ is acyclic, where we are considering paths (hence cycles) in the sense of directed graphs, always moving in the direction specified by arcs.

Now consider an arbitrary complete, simple, labelled digraph $G = ([n], E \sqcup E^*)$, where

$$E := \{(i, j) : i < j\},$$

$$E^* := \{(i, j) : i > j\}.$$

Given a subset $A \subseteq E \sqcup E^*$ of edges, we define the *transitive closure* \bar{A} of A in G to be the set of ordered pairs (i, j) of vertices which are joined by a path consisting of A -edges in G directed from i to j . The *transitive part* of this closure \bar{A} is defined to be

$$\mathcal{T}(A) := \bar{A} \setminus A$$

so that

$$\bar{A} = A \sqcup \mathcal{T}(A).$$

In particular, E and E^* are subsets of edges of G so we may consider their transitive closure in G . Note that E and E^* (equivalently G) coming from a permutation will be unchanged by this transitive closure operation, i.e. in this case we would have

$\mathcal{T}(E) = \emptyset = \mathcal{T}(E^*)$. The following is a trivial, but important, observation about taking transitive closures:

Lemma 9.1.1. *Given a subset A of edges of G , we have $\overline{\overline{A}} = \overline{A}$. Equivalently, $\mathcal{T}(\overline{A}) = \emptyset$.*

PROOF. Evidently $\overline{\overline{A}} \supseteq \overline{A}$. For the opposite containment, let $(i, j) \in \overline{\overline{A}}$. This means there is a path P consisting of edges $e_1, \dots, e_k \in \overline{A}$ directed from i to j (if $k = 1$, this means $(i, j) = e_1 \in \overline{A}$). Here, we have indexed the edges e_1, \dots, e_k in the order they appear in P . Namely, e_1 has initial vertex i and terminal vertex equal to the initial vertex of e_2 , and so on. Of course, e_k has terminal vertex j .

Note that each e_i is either an original edge of A , or else comes from a directed path P_i consisting of edges from A directed from the initial end to the terminal end of e_i . Hence, we can construct from P a path P' consisting only of A -edges in the following way: if $e_i \in A$, keep it; otherwise, replace e_i with the directed path P_i . Then P' is a directed path of A -edges from i to j , so $(i, j) \in \overline{A}$. \square

In other words, Lemma 9.1.1 says that taking the transitive closure of a set of edges produces a set of edges which is transitively closed. We are ready to give some equivalent criteria which guarantee that G is induced by a permutation:

Lemma 9.1.2. *The following are equivalent:*

- (i) $G = G(\omega)$ for some unique permutation $\omega \in S_n$.
- (ii) G is acyclic.

(iii) $E = \overline{E}$ and $E^* = \overline{E^*}$ (equivalently $\mathcal{T}(E) = \emptyset = \mathcal{T}(E^*)$).

PROOF. (i) \Rightarrow (ii). This is obvious, as all edges of $G(\omega)$ are directed from $\omega(i)$ to $\omega(j)$ for each $1 \leq i < j \leq n$.

(ii) \Rightarrow (i). Suppose G is acyclic. We claim that there exists a unique vertex $v_1 \in [n]$ such that all edges incident there are inwardly-directed. Indeed, if there were no such vertex then we could enter and leave every vertex, eventually constructing a cycle as G is finite; contradiction. We get uniqueness of v_1 since, for any other vertex $v \neq v_1$, G complete implies there is an edge directed from v to v_1 (v_1 has all inwardly-directed incident edges) so that v has an outwardly-directed incident edge.

Define $\omega(n) = v_1$, and delete v_1 from G , giving a new labelled, complete, simple digraph $G - \{v_1\}$ with vertex set $[n] \setminus \{v_1\}$. Of course $G - \{v_1\}$ is still acyclic, so we may repeat the above argument on this new digraph, giving a unique vertex $v_2 \in [n] \setminus \{v_1\}$ such that all edges incident there are inwardly-directed. We put $\omega(n-1) = v_2$ and continue in this way, finally arriving at a unique permutation $\omega \in S_n$ such that $G = G(\omega)$.

(ii) \Rightarrow (iii). Suppose, say, $E \neq \overline{E}$. Then there exists $(i, j) \in \overline{E} \setminus E$. Hence, we can find edges $e_1, \dots, e_k \in E$, $k > 1$, that form a directed path from i to j in G (i.e., the terminal end of e_t is the initial end of e_{t+1} for each $1 \leq t \leq k-1$). Since $(i, j) \notin E$ and G is complete, we have $(j, i) \in E^*$. Therefore $C := (e_1, \dots, e_k, (j, i))$ forms a cycle in G . By a similar argument we can show that $E^* \neq \overline{E^*}$ implies G contains a cycle.

(iii) \Rightarrow (ii). Suppose G contains a cycle. Since G is both antisymmetric and complete, it contains a cycle of length 3. Let a, b and c be the distinct vertices in $[n]$ that form

this cycle. Re-labelling if necessary, we may assume $a < b < c$. If the cycle is (a, b, c) , then

$$(a, b), (b, c) \in E; \quad (c, a) \in E^*$$

so that $(a, c) \in \overline{E} \setminus E$, i.e., $E \neq \overline{E}$. On the other hand, if (a, c, b) is the cycle, then

$$(a, c) \in E; \quad (c, b), (b, a) \in E^*$$

so that $(c, a) \in \overline{E^*} \setminus E^*$, i.e., $E^* \neq \overline{E^*}$. This completes the proof of Lemma 9.1.2. \square

9.2 Computing Infs and Sups in the Weak Order Lattice

With this machinery, we now show that the poset (S_n, \preceq) is a lattice. What's more, we can say precisely how to compute $\inf\{\pi_1, \dots, \pi_r\}$ ($\sup\{\pi_1, \dots, \pi_r\}$ resp.), where $\pi_1, \dots, \pi_r \in S_n$.

Lemma 9.2.1. *(S_n, \preceq) is a lattice with*

$$E(\inf\{\pi_1, \dots, \pi_r\}) = \overline{\cup_{i=1}^r E(\pi_i)}$$

and

$$E^*(\sup\{\pi_1, \dots, \pi_r\}) = \overline{\cup_{i=1}^r E^*(\pi_i)}.$$

PROOF. We will prove this only for infimums; the proof for supremums is completely analogous. By Lemma 9.1.2, it is sufficient to prove that the complete, simple, labelled digraph $G = ([n], E \sqcup E^*)$, where $E = \overline{\cup_{i=1}^r E(\pi_i)}$, contains no cycle.

Suppose G does contain a cycle. Then, since G is both antisymmetric and complete, it contains a cycle of length 3, passing through the vertices a, b and c , say. We may assume $a < b < c$; otherwise just re-label the vertices. If the cycle is (a, b, c) , then

$$(a, b), (b, c) \in E; \quad (c, a) \in E^*,$$

which violates the transitivity of E (note that E is transitively closed by Lemma 9.1.1). So this is impossible.

On the other hand, suppose the cycle is (a, c, b) . Then

$$(a, c) \in E; \quad (c, b), (b, a) \in E^*.$$

Therefore $(a, b), (b, c) \notin \cup_{i=1}^r E(\pi_i)$, and hence

$$(c, b), (b, a) \in \cap_{i=1}^r E^*(\pi_i).$$

From transitivity, $(c, a) \in \cap_{i=1}^r E^*(\pi_i)$, and therefore

$$(a, c) \notin \cup_{i=1}^r E(\pi_i).$$

So, as $(a, c) \in E$, there exist indices i_1, \dots, i_k and vertices $a = x_1, x_2, \dots, x_k, x_{k+1} = c$ with $x_j < x_{j+1}$, $x_j \neq b$ and

$$(x_j, x_{j+1}) \in E(\pi_{i_j}), \quad \forall j \leq k.$$

Let $1 \leq \ell \leq k$ be the index such that $x_\ell < b < x_{\ell+1}$. If it happens that $(b, x_\ell) \in E^*(\pi_{i_\ell})$, then as $(x_\ell, x_{\ell+1}) \in E(\pi_{i_\ell})$ we must have $(b, x_{\ell+1}) \in E(\pi_{i_\ell})$ by transitivity of the permutation π_{i_ℓ} . Hence $(b, x_{\ell+1}) \in E$, and since $(x_{\ell+1}, x_{\ell+2}) \in E$ we get $(b, x_{\ell+2}) \in E$ by transitivity of E . Using repeatedly the transitivity of E in this way, we eventually obtain $(b, c) \in E$, contradicting $(c, b) \in E^*$.

Hence, it must be that $(x_\ell, b) \in E(\pi_{i_\ell})$. So $(x_\ell, b) \in E$, and by the transitivity of E we have $(a, x_\ell) \in E$. Therefore, using transitivity once more, $(a, b) \in E$, contradicting $(b, a) \in E^*$. Therefore G must be acyclic, and hence (Lemma 9.1.2) $G = G(\pi)$ for some unique permutation $\pi \in S_n$. Finally, any permutation $\omega \in S_n$ that is a lower bound for all of π_1, \dots, π_r will have

$$E(\omega) \supseteq \cup_{i=1}^r E(\pi_i)$$

by definition of the weak order. Hence, since $E(\omega)$ is transitively closed, we have $E(\omega) \supseteq E$. We have just shown $E = E(\pi)$, and hence

$$E(\omega) \supseteq E(\pi) \supseteq \cup_{i=1}^r E(\pi_i)$$

so that $\omega \preceq \pi \preceq \pi_i$, $1 \leq i \leq r$. That is, $\pi = \inf\{\pi_1, \dots, \pi_r\}$ and we are done. □

9.3 Some Equivalent Criteria for $\inf\{\pi_1, \dots, \pi_r\} = 12 \cdots n$

Let $\mathcal{T}(\mathcal{E}_r)$ denote the transitive part of the closure of $\mathcal{E}_r := \cup_{\ell=1}^r E(\pi_\ell)$. Note that any pair $(i, k) \in \mathcal{T}(\mathcal{E}_r)$ has $k \geq i + 2$ since we must be able to find j with $i < j < k$. Hence, no pair $(i, i + 1)$, $1 \leq i \leq n - 1$, could possibly belong to $\mathcal{T}(\mathcal{E}_r)$. By Lemma 9.2.1,

$$E(\inf\{\pi_1, \dots, \pi_r\}) = \overline{\mathcal{E}_r} = \mathcal{E}_r \sqcup \mathcal{T}(\mathcal{E}_r).$$

So, if $\inf\{\pi_1, \dots, \pi_r\} = 12 \cdots n$, the unique minimum in this lattice, then every pair (i, j) with $i < j$ belongs to $E(\inf\{\pi_1, \dots, \pi_r\})$ and hence every pair $(i, i + 1)$, $1 \leq i \leq n - 1$, must belong to \mathcal{E}_r . Thus, choosing $\pi_1, \dots, \pi_r \in S_n$ independently and uniformly at random, we have proved the containment of events

$$\{\inf\{\pi_1, \dots, \pi_r\} = 12 \cdots n\} \subseteq \bigcap_{i=1}^{n-1} \{(i, i + 1) \in \cup_{\ell=1}^r E(\pi_\ell)\}.$$

But the event on the right is also sufficient for $\{\inf\{\pi_1, \dots, \pi_r\} = 12 \cdots n\}$! Indeed, if every pair $(i, i + 1)$, $1 \leq i \leq n - 1$, belongs to \mathcal{E}_r , then taking the transitive closure of this set gives us *every* pair (i, j) with $i < j$! We have therefore proved

$$\{\inf\{\pi_1, \dots, \pi_r\} = 12 \cdots n\} = \bigcap_{i=1}^{n-1} \{(i, i + 1) \in \cup_{\ell=1}^r E(\pi_\ell)\}. \quad (9.1)$$

We can take this a step further. Given $\omega \in S_n$, introduce the set of *descents* of ω :

$$D(\omega) := \{i : \omega(i) > \omega(i + 1)\}.$$

Consider the event on the right-hand side of (9.1). We have

$$\begin{aligned}
(i, i+1) \in \cup_{\ell=1}^r E(\pi_\ell) \forall i \in [n-1] &\iff \forall i \in [n-1], \exists \ell \in [r], (i, i+1) \in E(\pi_\ell) \\
&\iff \forall i \in [n-1], \exists \ell \in [r], i \notin D(\pi_\ell^{-1}) \\
&\iff \bigcap_{\ell=1}^r D(\pi_\ell^{-1}) = \emptyset.
\end{aligned} \tag{9.2}$$

Moreover, observe that

$$\begin{aligned}
i \in D(\inf\{\pi_1, \dots, \pi_r\}^{-1}) &\iff (i+1, i) \in E^*(\inf\{\pi_1, \dots, \pi_r\}) \\
&\iff (i, i+1) \notin E(\inf\{\pi_1, \dots, \pi_r\}) \\
&\iff (i, i+1) \notin E(\pi_j) \forall j \\
&\iff (i+1, i) \in E^*(\pi_j) \forall j \\
&\iff i \in D(\pi_j^{-1}) \forall j.
\end{aligned}$$

This shows that $D(\inf\{\pi_1, \dots, \pi_r\}^{-1}) = \bigcap_{\ell=1}^r D(\pi_\ell^{-1})$. Combining this with (9.1) and (9.2), we have therefore proved:

Lemma 9.3.1. *Let $\pi_1, \dots, \pi_r \in S_n$ be selected independently and uniformly at random, and let $P_n^{(r)} := P(\inf\{\pi_1, \dots, \pi_r\} = 12 \cdots n)$. Then*

$$\begin{aligned}
P_n^{(r)} &\stackrel{(a)}{=} P\left(\bigcap_{i=1}^{n-1} \{(i, i+1) \in \cup_{\ell=1}^r E(\pi_\ell)\}\right) \\
&\stackrel{(b)}{=} P\left(D(\inf\{\pi_1, \dots, \pi_r\}^{-1}) = \bigcap_{\ell=1}^r D(\pi_\ell^{-1}) = \emptyset\right). \quad \square
\end{aligned}$$

This allows us to instead study the probabilities (a) and (b), whichever happens to be convenient for us.

Given $\omega \in S_n$, let ω' denote $\omega = \omega(1) \cdots \omega(n)$ reversed in order, so that $\omega' = \omega(n) \cdots \omega(1)$, i.e. $\omega'(j) = \omega(n - j + 1)$, $1 \leq j \leq n$. For example, if $\omega = 45123$ then $\omega' = 32154$. It is trivial to check that

$$\inf\{\pi_1, \dots, \pi_r\} = \tau \iff \sup\{\pi'_1, \dots, \pi'_r\} = \tau'.$$

Indeed, this only requires the observation

$$\cup_{\ell=1}^r E^*(\pi'_\ell) = \{(j, i) : (i, j) \in \cup_{\ell=1}^r E(\pi_\ell)\}$$

followed by an application of Lemma 9.2.1. So we have

Lemma 9.3.2. *Let $\pi_1, \dots, \pi_r \in S_n$ be selected independently and uniformly at random. Then*

$$P_n^{(r)} = P(\inf\{\pi_1, \dots, \pi_r\} = 12 \cdots n) = P(\sup\{\pi_1, \dots, \pi_r\} = n(n-1) \cdots 1).$$

PROOF. We need only observe that $\pi_1, \dots, \pi_r \in S_n$ independent and uniformly random implies that the permutations π'_1, \dots, π'_r are as well. \square

Hence, when answering the question “How likely is it that r independent and uniformly random permutations have infimum (supremum resp.) equal to the unique minimum (maximum resp.)?”, Lemma 9.3.2 allows us to restrict our attention to infimums. We are now in a position to prove Theorem 1.4.2, part 1.

9.4 Submultiplicativity Again

We wish to prove the submultiplicativity of $P_n^{(r)}$ as a function of n , thus proving existence of

$$\lim_{n \rightarrow \infty} \sqrt[n]{P_n^{(r)}} = \inf_{n \geq 1} \sqrt[n]{P_n^{(r)}}$$

([43, p. 23, ex. 98] again). For this, we make use of Lemma 9.3.1.

Let π_1, \dots, π_r be independent and uniformly random permutations of $[n_1 + n_2]$. Introduce

$$\pi_i[1, 2, \dots, n_1], \quad 1 \leq i \leq r,$$

the permutation of $[n_1]$ left after deletion of the elements $n_1 + 1, n_1 + 2, \dots, n_1 + n_2$ from π_i . Similarly

$$\pi_i[n_1 + 1, n_1 + 2, \dots, n_1 + n_2], \quad 1 \leq i \leq r,$$

is the permutation of $\{n_1 + 1, n_1 + 2, \dots, n_1 + n_2\}$ left after deletion of the elements $1, 2, \dots, n_1$ from π_i . Then the permutations

$$\pi_1[1, \dots, n_1], \dots, \pi_r[1, \dots, n_1], \pi_1[n_1 + 1, \dots, n_1 + n_2], \dots, \pi_r[n_1 + 1, \dots, n_1 + n_2]$$

are all uniform on their respective sets of permutations, and are mutually independent.

By Lemma 9.3.1,

$$\inf\{\pi_1, \dots, \pi_r\} = 12 \cdots (n_1 + n_2) \iff (i, i + 1) \in \cup_{\ell=1}^r E(\pi_\ell), \quad 1 \leq i \leq n_1 + n_2 - 1,$$

and hence

$$\begin{aligned} \inf\{\pi_1, \dots, \pi_r\} = 12 \cdots (n_1 + n_2) \\ \implies (i, i + 1) \in \cup_{\ell=1}^r E(\pi_\ell[1, \dots, n_1]), \quad 1 \leq i \leq n_1 - 1 \\ \iff \inf\{\pi_1[1, \dots, n_1], \dots, \pi_r[1, \dots, n_1]\} = 12 \cdots n_1. \end{aligned}$$

Denote this first event by $\mathcal{E}_{n_1+n_2}$, and the last by \mathcal{E}_{n_1} . Thus we have proved the containment of events $\mathcal{E}_{n_1+n_2} \subseteq \mathcal{E}_{n_1}$. Similarly, we have

$$\begin{aligned} \inf\{\pi_1, \dots, \pi_r\} = 12 \cdots (n_1 + n_2) \\ \implies (i, i + 1) \in \cup_{\ell=1}^r E(\pi_\ell[n_1 + 1, \dots, n_1 + n_2]), \quad n_1 + 1 \leq i \leq n_1 + n_2 - 1 \\ \iff \inf\{\pi_1[n_1 + 1, \dots, n_1 + n_2], \dots, \pi_r[n_1 + 1, \dots, n_1 + n_2]\} \\ = (n_1 + 1)(n_1 + 2) \cdots (n_1 + n_2). \end{aligned}$$

Denote the last event by $\mathcal{E}_{n_2}^*$, so that we have the containment $\mathcal{E}_{n_1+n_2} \subseteq \mathcal{E}_{n_2}^*$. Consequently

$$\mathcal{E}_{n_1+n_2} \subseteq \mathcal{E}_{n_1} \cap \mathcal{E}_{n_2}^*,$$

and since the events on the right are independent, this implies $P_{n_1+n_2}^{(r)} \leq P_{n_1}^{(r)} P_{n_2}^{(r)}$. Of course, the rest of the statement follows from the (by now familiar) classical Fekete lemma concerning sub(super)multiplicative sequences [43, p. 23, ex. 98]. \square

9.5 Sharp Asymptotics of $P_n^{(r)}$

We are now ready to finish the proof of Theorem 1.4.2. The proof divides naturally into three steps. First, we will establish the exact formula

$$P_n^{(r)} = \sum_{k=0}^{n-1} (-1)^k \sum_{\substack{b_1, \dots, b_{n-k} \geq 1 \\ b_1 + \dots + b_{n-k} = n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r}. \quad (9.3)$$

which in turn facilitates computation of a bivariate generating function related to $P_n^{(r)}$. Finally, analytical techniques applied to a special case of this generating function yields the asymptotic result stated in Theorem 1.4.2:

$$P_n^{(r)} \sim -\frac{1}{z^* h_r'(z^*)} \frac{1}{(z^*)^n}, \quad r \geq 2, \quad n \rightarrow \infty,$$

where $z^* = z^*(r) \in (1, 2)$ is the unique (positive) root of the equation

$$h_r(z) := \sum_{j \geq 0} \frac{(-1)^j}{(j!)^r} z^j = 0$$

within the disk $|z| \leq 2$.

Specifically, we will use this exact formula for $P_n^{(r)}$ to show that

$$P_n^{(r)} = [z^n] \frac{1}{h_r(z)}, \quad r \geq 1, \quad (9.4)$$

followed by some asymptotic analysis. As a partial check, for $r = 1$ we obtain

$$P_n^{(1)} = [z^n] \frac{1}{e^{-z}} = \frac{1}{n!},$$

as we should! Also, we immediately see that for $r \geq 2$, the limit $\lim_{n \rightarrow \infty} \sqrt[n]{P_n^{(r)}}$, whose existence we established last section, equals $1/z^*$.

9.5.1 Step 1: An Exact Formula for $P_n^{(r)}$

Here, we establish formula (9.3). Notice that, if $\pi_1, \dots, \pi_r \in S_n$ are independent and uniformly random, then so are the n -permutations $\pi_1^{-1}, \dots, \pi_r^{-1}$. Hence, the probability (b) in Lemma 9.3.1 is the same as

$$P \left(\bigcap_{i=1}^r D(\omega_i) = \emptyset \right),$$

where $\omega_1, \dots, \omega_r \in S_n$ are independent and uniformly random. That is, we need to compute the probability that r independent and uniformly random permutations have no common descents.

Now, given $I \subseteq [n-1]$, let \mathcal{E}_I denote the event “ I belongs to $D(\omega_j)$, $1 \leq j \leq r$ ”. So \mathcal{E}_I is the event that I is common to all of the $D(\omega_j)$ ’s. Then, by Lemma 9.3.1,

$$1 - P_n^{(r)} = P \left(\bigcup_{i \in [n-1]} \mathcal{E}_{\{i\}} \right).$$

By the principle of inclusion-exclusion,

$$P \left(\bigcup_{i \in [n-1]} \mathcal{E}_{\{i\}} \right) = \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{\substack{I \subseteq [n-1] \\ |I|=k}} P \left(\bigcap_{i \in I} \mathcal{E}_{\{i\}} \right). \quad (9.5)$$

But notice that, given $I \subseteq [n-1]$,

$$\bigcap_{i \in I} \mathcal{E}_{\{i\}} = \mathcal{E}_I.$$

Hence, (9.5) becomes

$$P \left(\bigcup_{i \in [n-1]} \mathcal{E}_{\{i\}} \right) = \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{\substack{I \subseteq [n-1] \\ |I|=k}} P(\mathcal{E}_I). \quad (9.6)$$

So it only remains to compute $P(\mathcal{E}_I)$ for a fixed $I \subseteq [n-1]$, $|I| = k$, $k \in [n-1]$. This computation is an r -analog of the formula in Boná's book [11, pg. 4]. We present a modification of his argument.

Observe that

$$P(\mathcal{E}_I) = \frac{|\mathcal{E}_I|}{(n!)^r},$$

so we need to count the number of r -tuples $(\omega_1, \dots, \omega_r) \in \mathcal{E}_I$.

Write $I = \{i_1 < \dots < i_k\}$, and $J := [n-1] \setminus I = \{j_1 < \dots < j_{(n-1)-k}\}$. For $\omega \in S_n$, let $\bar{\omega}$ denote ω reversed in *rank*. So if $\omega = 45123$, then $\bar{\omega} = 21543$. Formally, $\bar{\omega}(j) = n - \omega(j) + 1$, $1 \leq j \leq n$. Notice that $D(\omega) \sqcup D(\bar{\omega}) = [n-1]$. Hence

$$D(\omega_j) \supseteq I, \quad 1 \leq j \leq r \quad \iff \quad D(\bar{\omega}_j) \subseteq J, \quad 1 \leq j \leq r.$$

Again, $\omega_1, \dots, \omega_r$ independent and uniformly random implies that so are the permutations $\bar{\omega}_1, \dots, \bar{\omega}_r$, so our task becomes to count the number of r -tuples of permutations (τ_1, \dots, τ_r) such that $D(\tau_j) \subseteq J$ for every j . As the τ_j are independent, this is just

$$|\{\omega \in S_n : D(\omega) \subseteq J\}|^r.$$

To count $|\{\omega \in S_n : D(\omega) \subseteq J\}|$, we arrange the n entries of ω into $n - k$ segments so that the first i segments together have j_i entries for each i . Then, within each segment, we put the entries into increasing order. Then the only places where the resulting ω could possibly have a descent is where two segments meet, i.e., at entries $j_1, \dots, j_{(n-1)-k}$, and hence $D(\omega) \subseteq J$.

The first segment of ω has to have length j_1 , and therefore can be chosen in $\binom{n}{j_1}$ ways. The second segment has to be of length $j_2 - j_1$, and must be disjoint from the first one, so may be chosen in $\binom{n-j_1}{j_2-j_1}$ ways. In general, segment i must have length $j_i - j_{i-1}$ if $1 < i < n - k$, and has to be chosen from the remaining $n - j_{i-1}$ entries, in $\binom{n-j_{i-1}}{j_i-j_{i-1}}$ ways. There is only one choice for the last segment, as all remaining $n - j_{(n-1)-k}$ entries must go there. Therefore

$$\begin{aligned} |\{\omega \in S_n : D(\omega) \subseteq J\}| &= \binom{n}{j_1} \binom{n-j_1}{j_2-j_1} \binom{n-j_2}{j_3-j_2} \cdots \binom{n-j_{(n-1)-k}}{n-j_{(n-1)-k}} \\ &= \frac{n!}{j_1!(j_2-j_1)! \cdots (n-j_{(n-1)-k})!}, \end{aligned}$$

and consequently

$$P(\mathcal{E}_I) = \frac{|\mathcal{E}_I|}{(n!)^r} = \frac{1}{j_1!(j_2-j_1)! \cdots (n-j_{(n-1)-k})!}.$$

Putting this into (9.6), we obtain

$$\begin{aligned}
1 - P_n^{(r)} &= P \left(\bigcup_{i \in [n-1]} \mathcal{E}_{\{i\}} \right) \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{\substack{I \subseteq [n-1] \\ |I|=k}} \frac{1}{j_1!^r (j_2 - j_1)!^r \cdots (n - j_{(n-1)-k})!^r} \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{\substack{b_1, \dots, b_{n-k} \geq 1 \\ b_1 + \dots + b_{n-k} = n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r},
\end{aligned}$$

where $b_1 = j_1$, $b_i = j_i - j_{i-1}$, $1 < i < n - k$, and $b_{n-k} = n - j_{(n-1)-k}$. This is clearly equivalent to (9.3).

9.5.2 Step 2: A Generating Function for $P_n^{(r)}$

Let us next use the formula (9.3) to establish the relation (9.4). Recall that we have defined $\mathcal{E}_{\{i\}}$ as the event “ i belongs to every $D(\pi_j^{-1})$, $1 \leq j \leq n - 1$ ”, and that

$$1 - P_n^{(r)} = P \left(\bigcup_{i=1}^{n-1} \mathcal{E}_{\{i\}} \right).$$

Introduce the random variable $S_n^{(r)} = S_n^{(r)}(\pi_1, \dots, \pi_r)$, the number of events $\mathcal{E}_{\{i\}}$ that are satisfied. As we have seen (Lemma 9.3.1), $S_n^{(r)}$ is also the number of descents in $\inf\{\pi_1, \dots, \pi_r\}^{-1}$. Formally, $S_n^{(r)}$ is the sum of indicators

$$S_n^{(r)} = \sum_{i=1}^{n-1} I_{\mathcal{E}_{\{i\}}}.$$

Observe that

$$P_n^{(r)} = P(S_n^{(r)} = 0),$$

so the formula (9.3) gives the probability $P(S_n^{(r)} = 0)$. But, in fact, this formula tells us even more about $S_n^{(r)}$. Indeed, consider the k -th (unsigned) term in this expression

$$\sum_{\substack{I \subseteq [n-1] \\ |I|=k}} P\left(\bigcap_{i \in I} \mathcal{E}_{\{i\}}\right) = \sum_{\substack{b_1, \dots, b_{n-k} \geq 1 \\ b_1 + \dots + b_{n-k} = n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r}.$$

This is the expected number of k -sets of the events $\mathcal{E}_{\{i\}}$ that occur simultaneously.

That is,

$$E\left[\binom{S_n^{(r)}}{k}\right] = \sum_{\substack{b_1, \dots, b_{n-k} \geq 1 \\ b_1 + \dots + b_{n-k} = n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r}, \quad 0 \leq k \leq n-1. \quad (9.7)$$

This produces the simple expression

$$P_n^{(r)} = \sum_{k=0}^{n-1} (-1)^k E\left[\binom{S_n^{(r)}}{k}\right].$$

We could have seen this another way, by observing that

$$P_n^{(r)} = P(S_n^{(r)} = 0) = E[(1-1)^{S_n^{(r)}}] = E\left[\sum_{k=0}^{n-1} (-1)^k \binom{S_n^{(r)}}{k}\right],$$

and using the linearity of expectation.

We will use these observations about $S_n^{(r)}$ to get a compact generating function related to this random variable, which happens to be amenable to asymptotic analysis.

Introduce the bivariate generating function

$$F_r(x, y) := \sum_{n \geq 1} x^n E \left[(1 + y)^{S_n^{(r)}} \right],$$

and let

$$f_r(z) := \sum_{\beta \geq 0} \frac{z^\beta}{(\beta + 1)!^r}.$$

Using what we know about $S_n^{(r)}$, we can simplify $F_r(x, y)$:

$$\begin{aligned} F_r(x, y) &= \sum_{n \geq 1} x^n E \left[(1 + y)^{S_n^{(r)}} \right] \\ &= \sum_{n \geq 1} x^n \sum_{k=0}^{n-1} y^k E \left[\binom{S_n^{(r)}}{k} \right] \\ &= \sum_{n \geq 1} x^n \sum_{k=0}^{n-1} y^k \sum_{\substack{b_1, \dots, b_{n-k} \geq 1 \\ b_1 + \dots + b_{n-k} = n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r} \\ &= \sum_{k \geq 0} (xy)^k \sum_{n > k} x^{n-k} \sum_{\substack{b_1, \dots, b_{n-k} \geq 1 \\ b_1 + \dots + b_{n-k} = n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r} \\ &= \sum_{k \geq 0} (xy)^k \sum_{\nu \geq 1} x^\nu \sum_{\substack{b_1, \dots, b_\nu \geq 1 \\ b_1 + \dots + b_\nu = \nu + k}} \frac{1}{(b_1!)^r \cdots (b_\nu!)^r} \\ &= \sum_{k \geq 0} (xy)^k \sum_{\nu \geq 1} x^\nu \sum_{\substack{\beta_1, \dots, \beta_\nu \geq 0 \\ \beta_1 + \dots + \beta_\nu = k}} \frac{1}{(\beta_1 + 1)!^r \cdots (\beta_\nu + 1)!^r} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k \geq 0} (xy)^k [z^k] \sum_{\nu \geq 1} (xf_r(z))^\nu \\
&= \sum_{k \geq 0} (xy)^k [z^k] \frac{xf_r(z)}{1 - xf_r(z)} \\
&= \frac{xf_r(xy)}{1 - xf_r(xy)} \\
&= \frac{1}{1 - xf_r(xy)} - 1.
\end{aligned}$$

Therefore

$$E \left[(1 + y)^{S_n^{(r)}} \right] = [x^n] \frac{1}{1 - xf_r(xy)}, \quad n \geq 1. \quad (9.8)$$

Plugging $y = -1$ into this expression, we obtain

$$\begin{aligned}
P_n^{(r)} = P(S_n^{(r)} = 0) &= E \left[(1 - 1)^{S_n^{(r)}} \right] = [x^n] \frac{1}{1 - xf_r(-x)} \\
&= [x^n] \frac{1}{h_r(x)}, \quad n \geq 1,
\end{aligned} \quad (9.9)$$

where $h_r(x) = \sum_{j \geq 0} ((-1)^j / (j!)^r) x^j$, and this is (9.4). It should be duly noted that this generating function is a special case of one found by Richard Stanley [45], but it is probably safe to say that he was unaware of any connection with the weak ordering.

9.5.3 Step 3: Asymptotics

We are about to finish the proof; all of the combinatorial insights are behind us, and only some asymptotic analysis remains. Armed with formula (9.9), our goal is to use Darboux's theorem [2] to estimate

$$[z^n] \frac{1}{h_r(z)}, \quad h_r(z) = \sum_{j \geq 0} \frac{(-1)^j}{(j!)^r} z^j, \quad r \geq 2.$$

First of all, notice that for $z > 0$ we have

$$1 - z < h_r(z) < 1 - z + z^2/(2!)^r.$$

Hence, we get

$$0 = 1 - (1) < h_r(1); \quad h_r(2) < 1 - (2) + (2)^2/(2!)^r \leq 0, \quad r \geq 2.$$

So $h_r(z) = 0$ has a root in $(1, 2)$ by the intermediate value theorem.

Now, consider the circle $|z| = u$, where $u > 1$ will be specified later. Let

$$g(z) = 1 - z, \quad G(z) = \sum_{j \geq 2} \frac{(-1)^j}{(j!)^r} z^j.$$

$g(z) = 0$ has a single root, of multiplicity 1, within the circle $|z| = u$. For $|z| = u$,

$$|g(z)| \geq \min_{t \in [0, 2\pi)} |1 - ue^{it}| = u - 1,$$

and

$$\begin{aligned} |G(z)| &\leq \frac{u^2}{2^r} \left(1 + \frac{u}{3^r} + \frac{u}{3^r} \frac{u}{4^r} + \cdots \right) \\ &\leq \frac{u^2}{2^r} \cdot \frac{1}{1 - \frac{u}{3^r}}, \quad u < 3^r. \end{aligned}$$

If we can find $u \in (1, 3^r)$ such that

$$u - 1 > \frac{\frac{u^2}{2^r}}{1 - \frac{u}{3^r}}, \quad (9.10)$$

then, by Rouché's theorem [48], $h_r(z) = g(z) + G(z)$ also has a unique, whence real positive, root z^* within the circle $|z| = u$. The inequality (9.10) is equivalent to

$$F(u) := u^2(2^{-r} + 3^{-r}) - u(1 + 3^{-r}) + 1 < 0.$$

$F(u)$ attains its minimum at

$$\bar{u} = \frac{1 + 3^{-r}}{2(2^{-r} + 3^{-r})} \in (1, 3^r),$$

and

$$F(\bar{u}) = 1 - \frac{(1 + 3^{-r})^2}{4(2^{-r} + 3^{-r})}.$$

For $r > 2$,

$$4(2^{-r} + 3^{-r}) \leq 8 \cdot 2^{-3} = 1,$$

and so $F(\bar{u}) < 0$ in this case, and we are done. Actually, notice that our choice of circle radius

$$|z| = \bar{u} = \frac{1 + 3^{-r}}{2(2^{-r} + 3^{-r})} \in (2, 3^r), \quad r > 2.$$

So we have proved $h_r(z) = 0$ has a unique (positive) root $z^* = z^*(r) \in (1, 2)$ within the disk $|z| \leq 2$, $r > 2$, which is what we wanted.

On the other hand, for $r = 2$,

$$F(\bar{u}) = 1 - \frac{(1 + 1/9)^2}{1 + 4/9} > 0,$$

so this case requires a bit more attention. Instead, consider

$$g(z) = 1 - z + \frac{z^2}{(2!)^2} - \frac{z^3}{(3!)^2}, \quad G(z) = \sum_{j \geq 4} \frac{(-1)^j}{(j!)^2} z^j,$$

and our strategy will be analogous to the above. First,

$$g'(z) = -1 + z/2 - z^2/12 = -\frac{(z-3)^2 + 3}{12} < 0, \quad z \in \mathbb{R},$$

so $g(z) = 0$ has one real root, z_1 . Since $g(1) = 2/9 > 0$ and $g(2) = -2/9 < 0$, we have $z_1 \in (1, 2)$.

Let $z_2 = a + ib$, $\bar{z}_2 = a - ib$ denote the two complex roots of $g(z) = 0$. Then (Vieta's relations [48])

$$2a + z_1 = 9, \quad (a^2 + b^2)z_1 = 36.$$

In particular

$$a = \frac{9 - z_1}{2} > 3.5,$$

hence $|z_2| = |\bar{z}_2| > 3.5$. So, if we can find $u \in (z_1, 3.5)$ with

$$|g(z)| > |G(z)|, \quad |z| = u,$$

we will be done once again by Rouché's theorem. For $|z| = u$,

$$\begin{aligned} |G(z)| &\leq \frac{u^4}{(4!)^2} \left(1 + \frac{u}{5^2} + \frac{u}{5^2} \frac{u}{6^2} + \cdots \right) \\ &\leq \frac{u^4}{(4!)^2} \cdot \frac{1}{1 - \frac{u}{5^2}}, \quad u < 5^2. \end{aligned} \tag{9.11}$$

Take $u = 2$. Let us show that

$$\min_{|z|=2} |g(z)| = |g(2)| = \frac{2}{9}.$$

To this end, we bound

$$\begin{aligned} |g(z)| &= \frac{1}{36} |(z - z_1)(z - z_2)(z - \bar{z}_2)| \\ &\geq \frac{1}{36} (2 - z_1) \min_{|z|=2} |z - z_2| |z - \bar{z}_2|. \end{aligned}$$

Setting $z = 2e^{it}$, we obtain

$$\begin{aligned} |z - z_2|^2 |z - \bar{z}_2|^2 &= [(2 \cos t - a)^2 + (2 \sin t - b)^2] \cdot [(2 \cos t - a)^2 + (2 \sin t + b)^2] \\ &= (4 - 4a \cos t + a^2 + b^2 - 4b \sin t)(4 - 4a \cos t + a^2 + b^2 + 4b \sin t) \\ &= (4 - 4a \cos t + a^2 + b^2)^2 - 16b^2 \sin^2 t \\ &:= F(t). \end{aligned}$$

Then

$$\begin{aligned} F'(t) &= 8a \sin t (4 - 4a \cos t + a^2 + b^2) - 32b^2 \sin t \cos t \\ &= 8 \sin t [a(4 + a^2 + b^2) - 4(a^2 + b^2) \cos t]. \end{aligned}$$

So $F'(t) = 0$ if and only if $t = 0, \pi$, since

$$\begin{aligned} \frac{a(4 + a^2 + b^2)}{4(a^2 + b^2)} &= \frac{a}{4} + \frac{a}{a^2 + b^2} \\ &= \frac{9 - z_1}{8} + \frac{z_1(9 - z_1)}{72} \\ &= \frac{81 - z_1^2}{72} > \frac{77}{72} > 1. \end{aligned}$$

This inequality also shows that $F'(t)$ always has the same sign as $\sin t$, hence $F'(t) > 0$ for $t \in (0, \pi)$ and $F'(t) < 0$ for $t \in (\pi, 2\pi)$. So $F(t)$ attains its minimum at $t = 0$, and consequently on $|z| = 2$

$$\begin{aligned} |g(z)| &\geq (2 - z_1)\sqrt{F(0)} = (2 - z_1)(4 - 4a + a^2 + b^2) \\ &= (2 - z_1)(2 - z_2)(2 - \bar{z}_2) \\ &= g(2) = \frac{2}{9}. \end{aligned}$$

Combining this with (9.11), we are done since

$$|g(z)| \geq \frac{2}{9} > \frac{\frac{2^4}{(4!)^2}}{1 - \frac{2}{5^2}} \geq |G(z)|, \quad |z| = 2. \quad \square$$