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On elliptic solutions of a coupled nonlinear Schrödinger system

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HIGHLIGHTS

• Elliptic solutions of the Manakov system constructed with real quasiperiod.
• Effective parametrization for classification of reality conditions.
• Loci of the auxiliary variables used in the integration are determined.
• Manakov soliton is recovered in the soliton limit.
• Small-wave-modulation limit satisfies linearized dispersion of planewaves.

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ABSTRACT

An explicit formula is obtained for single-phase bounded elliptic solutions of the Manakov system of integrable coupled nonlinear Schrödinger equations in terms of the Weierstrass sigma function with a real quasiperiod. The parametrization is effective in the sense that the reality conditions are completely characterized for each of the three possible couplings: focusing–focusing, defocusing–defocusing and focusing–defocusing. The Manakov soliton is recovered in the soliton limit and the small-wave-modulation limit is shown to satisfy the linearized dispersion relation of planewave solutions.

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1. Introduction

The Manakov system [1] of integrable coupled nonlinear Schrödinger equations is

\[
\begin{align*}
&i u_t + u_{xx} + 2(\sigma_1|u|^2 + \sigma_2|v|^2)u = 0, \\
&i v_t + v_{xx} + 2(\sigma_1|u|^2 + \sigma_2|v|^2)v = 0,
\end{align*}
\]

where \(u(t, x)\) and \(v(t, x)\) are the orthogonal components of the envelope of a rapidly varying complex field and \(\sigma_1, \sigma_2 = \pm 1\). Coupled systems of nonlinear Schrödinger equations arise in the context of polarized birefringent optical fibers [2], as well as in such apparently disparate fields as Bose–Einstein condensates and crowd dynamics [3]. Soliton solutions are well-known and many other phenomena associated with this system have been studied, including instabilities, homoclinic connections and wave-breaking [4–8]. However real quasiperiodic wavetrain solutions of the Manakov system provide more general models for applications than the solitons and homoclinic connections which are limits of these solutions. In particular, a goal of this paper is to establish a foundation for the subsequent study of the Whitham equations for slowly modulated single-phase wavetrains. It is likely that Riemann invariants of the Whitham equations are projections onto the complex plane of the simple branch points of the invariant trigonal spectral curve, based on results known for other integrable partial differential equations with simpler spectral curves, see [9] and the references therein. This paper studies the explicit underlying single-phase solutions and their relation to the algebraic invariants of the associated trigonal spectral curve.

Historically, the problem of constructing real N-phase wavetrains has been studied for other integrable equations such as the integrable scalar nonlinear Schrödinger equation [10–12] and the sine–Gordon equation [13,14]. The main difficulty in those constructions was determining the correct loci of the auxiliary spectral variables used in the integration. Kamchatnov [9,15] demonstrated an effective technique for integrating auxiliary variables for the single-phase elliptic solutions of many integrable wave equations. In this paper, the effective technique of Kamchatnov is extended to the Manakov system (1) and the problem of integrating the auxiliary variables for single-phase wavetrains with real quasiperiod is solved.

The construction of elliptic solutions of the Manakov system has also been studied previously. Large classes of elliptic solutions...
have been constructed using a conventional amplitude–phase ansatz [16–18] but these results do not study the relation between the elliptic solutions of the single-phase ansatz of the Lax pair and the invariant trigonal spectral curve. In [19] the squared eigenfunctions of the Lax pair are used to obtain formulas for solutions in terms of the Weierstrass sigma function which are consistent with those obtained in the present paper. However, unlike the present paper, the parametrization of the solutions in [19] does not utilize the close connection between the invariant trigonal spectral curve of the coupled system and the simpler two-sheeted spectral curve of the scalar equation; consequently a completely explicit characterization of all reality conditions and limiting cases is not obtained in [19] because of the complexity of the parametrization, although several numerical examples are given of how the reality conditions may be satisfied. Recent results [20] for N-phase solutions of the focusing–focusing Manakov system use Riemann theta functions on the trigonal spectral curve to construct some of the single-phase elliptic solutions but the Manakov soliton is not found in the soliton limit of the single-phase solutions. In this paper, the complete collection of single-phase bounded elliptic solutions with real quasiperiod arising from the single-phase polynomial ansatz for the solution of the Lax pair is constructed, both the soliton limit and the small–wave–modulation limit are studied and, in particular, the Manakov soliton is shown to arise in the soliton limit. The apparent discrepancy with [20] is resolved by the fact that the deck transformations of the three-sheeted branched covering of the Riemann sphere associated with the real quasiperiodic wavetrains close to the Manakov soliton limit are different than the deck transformations considered in [20].

In summary, the present paper is concerned with the specific class of single-phase solutions coming from the lowest possible polynomial degree ansatz for the solution of the Lax pair, with the motivation of understanding the connection between the algebraic invariants of the trigonal spectral curve and the wavetrain. The results generalize the method used by Kamchatnov [15] for the effective integration of the scalar integrable nonlinear Schrödinger equation. In particular, the allowed values of all the parameters in all the solutions satisfying the reality conditions are explicitly defined and both the soliton limit and small–wave–modulation limit are studied in detail. The solutions for u and v are linear combinations of quasiperiodic functions, while the quantity \( \sigma_1|u|^2 + \sigma_2|v|^2 \) is periodic. The results apply to all three versions of the Manakov system, \( \sigma_1 = \sigma_2 = 1 \), corresponding to a focusing–focusing coupling, \( \sigma_1 = -\sigma_2 = 1 \), corresponding to a defocusing–defocusing coupling, and \( \sigma_1 = 1, \sigma_2 = -1 \), corresponding to a mixed focusing–defocusing coupling.

The integrability of system (1) is established through the equivalence of system (1) and the commutation of a Lax pair of linear eigenvalue problems, \( \psi_x = U \psi, \psi_t = V \psi \),

\[
\begin{align*}
U &= \begin{pmatrix} -2i\lambda & iu & iv \\ i\sigma_1u^* & iu & 0 \\ i\sigma_2v^* & 0 & i\lambda \end{pmatrix}, \\
V &= \begin{pmatrix} -6i\lambda^2 + i\sigma_2|u|^2 + i\sigma_3|v|^2 & 3iu\lambda - ui - i\sigma_1|u|^2 & 3iv\lambda + vi \\
3\sigma_2u^*\lambda + i\sigma_3|v|^2 & 3\lambda^2 - i\sigma_3|u|^2 & -i\sigma_1uv^* \\
3i\sigma_2v^*\lambda + i\sigma_3|u|^2 & -i\sigma_3uv^* & 3\lambda^2 - i\sigma_2|v|^2 \end{pmatrix},
\end{align*}
\]

where \( u^*, v^* \), denote the complex conjugates of \( u, v \), respectively, and \( \lambda \) is the spectral parameter in the inverse spectral theory of the integrable system.

In order to obtain single-phase solutions in terms of elliptic functions (or, in general, N-phase solutions) it is convenient to place the Lax pair in matrix commutator or squared eigenfunction form. In particular the Manakov system (1) is equivalent to the consistency, i.e., existence of a sufficiently large class of common solutions, of

\[
\psi_x = [U, \psi], \quad \psi_t = [V, \psi],
\]

where \([A, B] = AB - BA\) is the usual operation of matrix commutation. The matrix commutator formulation of the Lax pair (4) is equivalent to the squared eigenfunction equations for the scalar equation [9,19]. The solution matrix \( \Psi \) of Eq. (4) has constant trace and the commutation operator on the right-hand side of Eq. (4) implies that constant multiples of the identity matrix can be added to \( \Psi \) without loss of generality, so that the trace of \( \Psi \) can be assumed to be zero. Thus the determinant of \( \Psi \) is constant. If a polynomial ansatz in the spectral parameter \( \lambda \) is assumed for \( \Psi \), then an invariant algebraic curve is associated with the solutions of the Manakov system (1). In particular, the constant determinant of \( \Psi \) provides a set of integrals of motion which can be used to solve the Manakov system (1) explicitly. To obtain single-phase solutions of the Manakov system (1), a quadratic dependence on the spectral parameter \( \lambda \) is posited for \( \Psi \). The resulting class of elliptic solutions given in Eq. (76) contains, in the soliton limit studied in Section 6.1.1, the simplest non-trivial solution of the focusing–focusing system, viz., the Manakov soliton [1].

\[
u(t, x) = 2bc \text{sech}(2b(x + 4at)) \exp(-2i\alpha - 4i(a^2 - b^2)t),
\]

where \( a, b \in \mathbb{R} \) and \( c_1 \) and \( c_2 \) are arbitrary complex constants such that \( |c_1|^2 + |c_2|^2 = 1 \). Therefore a solution to Eq. (4) is constructed of the form,

\[
\Psi = \begin{pmatrix} \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} \end{pmatrix},
\]

where each entry in \( \Psi \) is a polynomial in \( \lambda \) of degree at most two. Upon substitution of this ansatz into Eq. (4), certain constraints are forced on the entries of \( \Psi \) so that all the differential equations satisfied by the entries of \( \Psi \) are mutually consistent with respect to complex conjugation. In addition, solutions are sought which are functions of a single real phase, which also places some restrictions on the entries \( \Psi_{22}, \Psi_{23}, \Psi_{24} \). In summary, with this ansatz, the entries of the solution matrix \( \Psi \) have the form,

\[
\begin{align*}
\Psi_{22} &= i\lambda^2 + i\alpha x - i/3 |\sigma_1u|^2 + i/3 , \\
\Psi_{23} &= i\lambda^2 + i\alpha x - i/3 |\sigma_2v|^2 + i/3 , \\
\Psi_{24} &= -i/3 |\sigma_1u|^2 v, \\
\Psi_{31} &= -i/3 |\sigma_2v|^2 u, \\
\Psi_{32} &= i/3 |\sigma_1u|^2 v, \\
\Psi_{33} &= i/3 |\sigma_2v|^2 u ,
\end{align*}
\]

where \( f_1, \alpha \in \mathbb{R} \) are real constants and \( \mu_1 \) and \( \mu_2 \) are the auxiliary variables whose solutions provide an intermediate step to solving for \( u \) and \( v \). It is worth pointing out that if the constant \( \alpha \) is replaced by two distinct constants \( \alpha_1 \neq \alpha_2 \) in \( \Psi_{22} \) and \( \Psi_{23} \), then the solutions to the Manakov system which arise are not expressible in terms of elliptic functions (functions on a genus-one Riemann surface) but instead require ultra–elliptic functions (functions on a genus-two Riemann surface) [21].

The commutator Eq. (4) can be written explicitly as a set of eight coupled equations for \( u, v, \lambda, \mu_1, \mu_2 \),

\[
\begin{align*}
\mu_{1x} &= -3i|\mu_1|^2 - 3if_1 \mu_1 + 2i/3 (|\sigma_1|u|^2 + |\sigma_2|v|^2) - 3i\alpha x , \\
\mu_{2x} &= -3i|\mu_2|^2 - 3if_1 \mu_2 + 2i/3 (|\sigma_1|u|^2 + |\sigma_2|v|^2) - 3i\alpha x ,
\end{align*}
\]
\[ u_t = 3\nu \mu_1 + 3i\nu f_1, \]
\[ v_t = 3i\nu \mu_2 + 3i\nu f_1, \]
\[ \mu_{1t} = -3f_1 \mu_{1x}, \]
\[ \mu_{2t} = -3f_1 \mu_{2x}, \]
\[ u_t = -3f_1 u_x + 9i\alpha u, \]
\[ v_t = -3f_1 v_x + 9i\alpha v. \]

Making the change of variables to a single-phase variable \( \xi = x - 3f_1 t + \xi_0 \) in (8) leads to the solutions,
\[ \mu_1 = \tilde{\mu}_1(\xi), \quad \mu_2 = \tilde{\mu}_2(\xi), \quad u = e^{9i\alpha t} \tilde{u}(\xi), \quad v = e^{9i\alpha t} \tilde{v}(\xi), \]
where \( \tilde{\mu}_1, \tilde{\mu}_2, \tilde{u} \) and \( \tilde{v} \) satisfy the x-flow equations of (8) with \( \xi \) in the place of x. In the following, the x-flow equations of (8) are solved, with the substitutions (9) made only at the end of the calculation in order to obtain the full solution. The equations for the evolution of \(|u|^2\) and \(|v|^2\) are also of use in solving the system (8),
\[ |u|^2 = 3i|u|^2(\mu_1 - \mu_2^*), \quad |v|^2 = 3i|v|^2(\mu_2 - \mu_1^*). \]

Although, the differential equations in (8) are identical for \( \mu_1 \) and \( \mu_2 \), the initial conditions are not arbitrary. The key to the algebraic invariant approach to finding explicit solutions of integrable equations used in [9,15], is that the invariant spectral curve of genus one is not arbitrary but must satisfy the reality conditions determined by the boundedness of the solutions and the reality of their quasiperiod.

2. Algebraic invariants

The invariant spectral curve \( C \) of genus one is a trigonal curve
\[ \det(wI - \Psi) = w^3 + A(\lambda)w + B(\lambda) = 0, \]
where
\[ A(\lambda) = 3\lambda^4 + 6f_1 \lambda^3 + (6\alpha + 3f_2^2)\lambda^2 \]
\[ + (6\alpha^2 - \sigma_1 \lambda_1 - \sigma_2 \lambda_2) \lambda + 3\alpha^2 + I_3, \]
\[ B(\lambda) = -2\alpha^6 - 6\alpha f_1 \lambda^3 - 6(\alpha + f_2^2)\lambda^2 \]
\[ - i(12f_1 \alpha + 2f_3^2 - \sigma_1 \lambda_1 - \sigma_2 \lambda_2) \lambda^3 \]
\[ - i(6\alpha^2 + 6f_1^2 \alpha - \sigma_1 f_1 - \sigma_2 f_1 + I_3) \lambda^2 \]
\[ - i(6\alpha^2 - \sigma_1 \alpha \lambda_1 - \sigma_2 \alpha \lambda_2 + f_1 \lambda_1) \lambda \]
\[ - i(2\alpha^2 + \alpha \lambda_1 - \frac{1}{3} \sigma_1 \sigma_2 \lambda_4). \]

are written in terms of the following four integrals of motion,
\[ I_1 = |u|^2(\mu_1 + \mu_2^* + f_1), \]
\[ I_2 = |v|^2(\mu_2 + \mu_1^* + f_1), \]
\[ I_3 = \sigma_1 |u|^2|\mu_1|^2 + \sigma_2 |v|^2|\mu_2|^2 + \frac{1}{9}(\sigma_1 |u|^2 + \sigma_2 |v|^2)^2 \]
\[ - \sigma(\sigma_1 |u|^2 + \sigma_2 |v|^2), \]
\[ I_4 = |u|^2|v|^2(\mu_1 - \mu_2)^2. \]

The spectral curve \( C \) can also be written in the form,
\[ (w + 2i\alpha^2 + 2i\lambda_1 \lambda_2 + 2\alpha \lambda_1 + 2\lambda_1 \lambda_2) (w - i\lambda_1^2 - i\lambda_2^2 + i\sigma_1 \lambda_1 - i\alpha)^2 \]
\[ + (I_3 - (\sigma_1 \lambda_1 + \sigma_2 \lambda_2) \lambda)(w - i\lambda_1^2 - i\sigma_1 \lambda_1 - i\alpha) \]
\[ + \frac{1}{3} i\sigma_1 \sigma_2 \lambda_4 = 0. \]

Notice that the invariance of the spectral curve \( C \) implies the invariance of \( S = \sigma_1 \lambda_1 + \sigma_2 \lambda_2 \), as well as \( I_1 \) and \( I_4 \). However, the individual invariance of \( I_1 \) and \( I_2 \) can be verified directly from the differential equation (8). Similarly the quantity
\[ K = uv(\mu_1 - \mu_2), \]
where \( I_4 = |K|^2 \), satisfies a simple evolution equation obtained by direct calculation from Eq. (8) with solution
\[ K = K_0 e^{3i\theta x}, \]
where \( K_0 = u(0)v(0)(\mu_1(0) - \mu_2(0)) \) is constant and \( I_4 = |K_0|^2 \).

The three-sheeted Riemann surface associated with the algebraic curve \( C \) is generically of genus one, since the discriminant of the third-degree polynomial in \( w \) is of degree six in \( \lambda \), implying that the three-sheeted curve has, in general, six simple branch points above the complex plane, and the branched covering of the Riemann sphere is unramified above the point at infinity, so the Riemann surface is a topological torus. There exists, therefore, a birational transformation to the standard Weierstrass form of an elliptic curve, compare [20] for a similar formula, such that if the expressions in Box 1 hold, then \( X \) and \( Y \) satisfy the equation of an elliptic curve \( C \),
\[ Y^2 = \omega(X) = 4X^3 - g_2X - g_3 \]
where
\[ g_2 = 108 \left(\alpha - \frac{1}{4}f_1^2\right)^2 + 18f_1 s + 36I_3, \]
\[ g_3 = 36\sigma_1 \sigma_2 I_4 \]
and
\[ g_2 = 108 \left(\alpha - \frac{1}{4}f_1^2\right)^3 - 3 \left(\alpha - \frac{1}{4}f_1^2\right) g_2 + 9s^2. \]

For completeness, the inverse transformation is
\[ X = -3i(w - i\lambda^2 - if_1 \lambda - i\alpha) + 3 \left(\alpha - \frac{1}{4}f_1^2\right). \]

The discriminant of the curve \( C \) is a sixth degree polynomial \( D = 4A(\lambda)^3 + 27B(\lambda)^2 \) in \( \lambda \), where the polynomials \( A(\lambda) \) and \( B(\lambda) \) were defined in Eq. (12). In the special case where \( I_4 = 0 \), i.e., the case in which the Manakov system degenerates to a pair of scalar cubic nonlinear Schrödinger equations, the sextic polynomial \( D \) factors with a repeated linear factor \( F_1 \) and a quartic factor \( F_4 \), the quartic being reduced when written in terms of the variable \( z = \lambda + \frac{1}{2}f_1, \) viz.,
\[ F_1 = Sz - \frac{1}{36} g_2 + 3\alpha^2 - \frac{3}{2} of_1^2 + \frac{3}{16} f_1^4, \]
\[ F_4 = z^4 + 2 \left(\alpha - \frac{1}{4}f_1^2\right) z^2 - 4 \frac{g_2}{9}, \]
\[ + \frac{1}{81} g_2 - \frac{1}{3} \left(\alpha - \frac{1}{4}f_1^2\right)^2. \]
Following the procedure of seeking a resolvent polynomial for the reduced quartic \( F_4 \), see [9,22], a factorization into two quadratic factors is sought,
\[ F_4 = (z^2 + mz + n)(z^2 + sz + t), \]
in which \( -m \) is the sum of two of the roots of the reduced quartic \( F_4 \). The factorization is possible if and only if \( m^2 \) is a root of a resolvent cubic equation of the reduced quartic \( F_4 \), viz.,
\[ m^2 = \frac{4}{9} v - \frac{4}{3} \left(\alpha - \frac{1}{4}f_1^2\right). \]
\[ \text{where} \ v \text{ is a root of the cubic equation associated with the Weierstrass form of the elliptic spectral curve } \tilde{C}_0 \text{ given in Eq. (21) with} \]
\[
\lambda = \frac{4Y + 12(\sigma_1 I_1 + \sigma_2 I_2) + 12f_1 X + 9f_1^2 - 36f_1 \alpha}{6(-4X - 3f_1^2 + 12\alpha)},
\]
\[
w = \frac{4(4Y + 3(\sigma_1 I_1 + \sigma_2 I_2))^2 + 48X^3 + (72f_1^2 - 288\alpha)X^2 - (216f_1^2 \alpha - 27f_1^4 - 432\alpha^2)X}{(-4X - 3f_1^2 + 12\alpha)^2}.
\]

Box 1.

\[I_4 = 0,\]
\[4v^3 - g_2v - g_30 = 4(v - e_1)(v - e_2)(v - e_3) = 0. \tag{27}\]

Letting \(m_i, i = 1, \ldots, 6\), be the six solutions for \(m\) and \(z_i, i = 1, \ldots, 4\), be the four roots of \(F_4\), using \(m_1 = -z_1 - z_2, m_2 = -m_1, \) etc., and repeatedly using the fact that \(z_1 + z_2 + z_3 + z_4 = 0\), the three roots \(e_1, e_2, e_3\), of the cubic (27) can be written in terms of the four \(\lambda\) roots of the discriminant of the spectral curve associated with the four roots of \(F_2\) in a simple fashion, similar to the expressions found in [9,15],

\[
e_1 = \frac{9}{16}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2 + 3 \left(\alpha - \frac{1}{4}f_1^2\right),
\]
\[
e_2 = \frac{9}{16}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)^2 + 3 \left(\alpha - \frac{1}{4}f_1^2\right),
\]
\[
e_3 = \frac{9}{16}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)^2 + 3 \left(\alpha - \frac{1}{4}f_1^2\right).
\]

Also, the coefficients of \(F_4\) in (25) can be expressed in terms of symmetric functions of the zeros of \(F_4\), which leads to the expressions,

\[
e_1 = \frac{3}{4}(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_4) + (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3),
\]
\[
e_2 = \frac{3}{4}(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) + (\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2),
\]
\[
e_3 = \frac{3}{4}(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_3) + (\lambda_1 - \lambda_3)(\lambda_4 - \lambda_2),
\]

and

\[
g_2 = \frac{81}{16}(\lambda_1 - \lambda_2)^2(\lambda_3 - \lambda_4)^2 + \frac{27}{16}((\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4) + (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3))^2.
\]

Eqs. (28)-(30) are only valid when \(I_4 = 0\), however these expressions will be useful in Section 5 for characterizing the reality conditions for solutions in the fully coupled case where \(I_4 > 0\), because Eq. (22) shows that \(g_2\) is independent of \(I_4\) and \(g_3\) depends linearly on \(I_4\), so the relation between \(C\) and \(C_0\) is a simple vertical translation of the cubic polynomial.

3. Auxiliary variables

The auxiliary variables \(\mu_1\) and \(\mu_2\) contain information about the relative intensities of the two channels of the coupled wave, i.e., the internal polarization of the coupled waveform. In particular, the polarization ratio

\[
\frac{|u|^2}{|v|^2} = \frac{I_1(\mu_2 + \mu^*_1 + f_1)}{I_2(\mu_1 + \mu^*_1 + f_1)} \tag{31}
\]

shows that, in the simplest case of \(f_1 = 0\), the ratio of \(|u|^2\) to \(|v|^2\) is proportional to the ratio of the real part of \(\mu_2\) to the real part of \(\mu_1\). In other words, the ratio of the real parts of the auxiliary variables \(\mu_1\) and \(\mu_2\) has a direct physical interpretation in terms of the distribution of energy between the two channels of the coupled waveform. The imaginary parts of \(\mu_1\) and \(\mu_2\) are related to the evolution of the polarization by

\[
\frac{d}{dx} \ln \left(\frac{|u|^2}{|v|^2}\right) = 3i(\mu_1 - \mu_1^*) - (\mu_2 - \mu_2^*)), \tag{32}
\]

so that the imaginary parts of \(\mu_1\) and \(\mu_2\) are identically equal if and only if the polarization in Eq. (31) is constant.

If \(I_4 = 0\), then either \(|u| = 0\) or \(|v| = 0\) or \(\mu_1 = \mu_2\) (which by Eqs. (31) and (32) implies that the polarization is constant), so that the problem of solving the coupled system reduces to that of solving the scalar integrable NLS equation. Therefore it is assumed, without loss of generality, that \(I_4 > 0\) and, consequently, \(|u| = |v| \neq 0\) and \(\mu_1 \neq \mu_2\). Moreover, Eq. (32) shows that the evolution of the polarization (31) is non-trivial precisely when \(\mu_1 \neq \mu_2\). In this case, the assumption \(\mu_1 \neq \mu_2\) can be strengthened to the inequality of the imaginary parts of \(\mu_1\) and \(\mu_2\).

The solutions \(u\) and \(v\) can be constructed explicitly by first solving for the auxiliary variables \(\mu_1\) and \(\mu_2\), following the procedure for the scalar equation [9,15]. Using Eqs. (13), (14) and (16), a quadratic equation for \(\mu_1 - \mu_2\) can be obtained. The solution of the quadratic equation provides the relationship between the two auxiliary variables,

\[
\mu_1 = \mu_2 + \frac{1}{2} \left(\frac{I_1}{|u|^2} - \frac{I_2}{|v|^2}\right) \pm \frac{i \sqrt{\varphi}}{2|u|^2|v|^2}, \tag{33}
\]

where

\[
\varphi = -(I_1|v|^2 - I_2|u|^2)^2 + 4I_4|u|^2|v|^2
\]

\[
= 4|u|^2|v|^2(\mu_1 - \mu_2)^2 > 0,
\]

where the strict inequality follows from the discussion in the previous paragraph.

Eqs. (15) and (33) are used to obtain a quadratic equation for \(\mu_2\) which can be solved to find \(\mu_2\). Then Eq. (33) is used to obtain \(\mu_1\). Hence

\[
\mu_1 = \frac{I_1}{2|u|^2} - \frac{f_1}{2} \pm \frac{ia_1 \sqrt{\varphi}}{6(\sigma_1 |u|^2 + \sigma_2 |v|^2)},
\]

\[
\mu_2 = \frac{I_2}{2|v|^2} - \frac{f_1}{2} \pm \frac{ia_1 \sqrt{\varphi}}{6(\sigma_1 |u|^2 + \sigma_2 |v|^2)},
\]

where

\[
v = -\sigma_1 |u|^2 - \sigma_2 |v|^2 + 3 \left(\alpha - \frac{1}{4}f_1^2\right) \tag{36}
\]

and \(\varphi\) is the cubic polynomial defined by the Weierstrass form of the spectral curve (21).

Eq. (35) is valid for all \(x\) for which \(\sigma_1 |u|^2 + \sigma_2 |v|^2 \neq 0\). In case \(\sigma_1 \sigma_2 = 1, \sigma_1 |u|^2 + \sigma_2 |v|^2 \neq 0\) for all \(x\) since \(I_4 > 0\) implies \(|u| \neq 0\) and \(|v| \neq 0\). In case \(\sigma_1 \sigma_2 = -1, \) a non-trivial solution must have \(|u|^2 \neq |v|^2\) on some interval of \(x\) values. The boundedness of the
Eq. (35) shows that $\wp$ which can be solved in terms of the Weierstrass $\mu$ in $\sigma$ have opposite signs and the second square roots have the same sign in $\mu_1$ and $\mu_2$. Moreover $\wp(\nu) \geq 0$ since the real parts of $\mu_1$ and $\mu_2$ are known from Eqs. (13) and (14), and are correctly given by the first and second terms in Eq. (35).

Direct calculation of the derivative of the real variable $\nu$ given in Eq. (36) using Eq. (10) and the explicit expressions for $\mu_1$ and $\mu_2$ in Eq. (35), shows that $\nu$ satisfies the following differential equation which can be solved in terms of the Weierstrass $\wp$ function,

$$\frac{d\nu}{dx} = \pm \sqrt{\wp(\nu)}. \quad (37)$$

The reality condition, viz., the solutions for the real variable $\nu$ must be bounded and have a real period, implies $\wp(\nu) = 4(\nu - e_1)(\nu - e_2)(\nu - e_3)$, where the three roots are real, distinct and ordered $e_3 < e_2 < e_1$, with $e_1 \leq \nu \leq e_2$, since $\wp(\nu) \geq 0$. The reality condition on the roots $e_1$, $e_2$, $e_3$ is not satisfied by all choices of the real integration constants $\alpha$, $f_1$, $l_1$, $l_2$, $l_3$, $l_4$. These reality conditions will be fully characterized in Section 5.

Assuming that initially $\nu(0) = e_3$, the solution for $\nu$ is expressible in terms of either Weierstrass or Jacobi elliptic functions as

$$\nu(x) = \wp(x + \omega') = e_3 + (e_2 - e_3)\sin^2(\sqrt{\nu_1 - e_3} x, k), \quad (38)$$

where $\omega'$ is the purely imaginary half-period of $\wp$ such that $\wp'(\omega') = e_3$ and the elliptic modulus is

$$k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}. \quad (39)$$

Thus Eqs. (36) and (38) give expressions for the signed sum of the amplitudes of the solutions for $\nu$ and $\nu_1$ in terms of elliptic functions,

$$\sigma_1|\nu|^2 + \sigma_2|\nu_1|^2 = 3 \left( \alpha - \frac{1}{4} \right) - \wp(x + \omega'),$$

$$= 3 \left( \alpha - \frac{1}{4} \right) - e_3 - (e_2 - e_3)\sin^2(\sqrt{\nu_1 - e_3} x, k). \quad (40)$$

Complete definitions and notational conventions for the elliptic and theta functions used in this paper are given in Appendix.

It is now necessary to eliminate the quantity $\nu$ from the Eq. (35) for the auxiliary variables. Notice that Eq. (18) gives an alternative expression for $\mu_1 - \mu_2$ to that given by Eq. (33). The consistency of these two expressions implies that

$$i\sqrt{\nu} = 2u\nu'K - l_1|\nu|^2 + l_2|\nu_1|^2. \quad (41)$$

Thus alternative expressions for $\mu_1$ and $\mu_2$ are obtained,

$$\mu_1 = \frac{\sigma_1 - \sigma_2}{2(\sigma_1|\nu|^2 + \sigma_2|\nu_1|^2)} - \frac{f_1}{2} + \frac{\sigma_2K}{\wp(\nu)} u^* \quad (42)$$

$$\mu_2 = \frac{\sigma_1 - \sigma_2}{2(\sigma_1|\nu|^2 + \sigma_2|\nu_1|^2)} - \frac{f_1}{2} + \frac{\sigma_1K}{\wp(\nu)} u^*$$

$$\pm \frac{\sigma_2K}{6(\sigma_1|\nu|^2 + \sigma_2|\nu_1|^2)}.$$
where $\sigma$ and $\zeta$ are the usual Weierstrass functions, see Appendix for definitions. It is convenient to introduce the quantity

$$
\Sigma = \frac{\sigma(k - x - \omega')\sigma(k + \omega')}{\sigma(k + x + \omega')\sigma(k - \omega')} e^{2i\pi k x} 
$$

and note that the integral in Eq. (52) is purely imaginary so that Eqs. (52) and (53) imply that $\Sigma^* = \Sigma^{-1}$. Hence $\Sigma$ is a quasi-periodic function lying on the unit circle in the complex plane.

Solving for $Q$, substituting into Eq. (45) and using the definition of $K$ in Eq. (19), gives

$$
u^* = \frac{-\sigma_1 l_1 + \sigma_2 l_2}{2\sigma_1 K} + \left(\frac{1 + \rho_1 e^{\rho_1} \Sigma}{1 - \rho_1 e^{\rho_1} \Sigma}\right) \frac{\sqrt{\Delta}}{2\sigma_2 K}.
$$

A similar calculation gives

$$w^* = \frac{\sigma_1 l_1 + \sigma_2 l_2}{2\sigma_1 K} - \left(\frac{1 + \rho_1 e^{\rho_1} \Sigma}{1 - \rho_1 e^{\rho_1} \Sigma}\right) \frac{\sqrt{\Delta}}{2\sigma_2 K}.
$$

The circle constraint (43) implies that $\nu^*$ and $w^*$ lie on bounded loci if and only if $l_1 l_2 \neq 0$. From the solutions in Eqs. (55) and (56), $l_1 l_2 \neq 0$ is equivalent to $\rho_1 \neq 1$ and $\rho_2 \neq 1$. The relation between the constants of integration $\rho_1, \rho_2, \theta_1$, and $\theta_2$ can be obtained by the consistency of Eqs. (55) and (56),

$$\rho_1 \rho_2 e^{i(\theta_1 - \theta_2)} = \frac{S - \sqrt{\Delta}}{S + \sqrt{\Delta}} \in \mathbb{R},
$$

where

$$\theta_2 = \begin{cases} \theta_1 + \pi & \text{if } \sigma_1 \sigma_2 = 1, \\ \theta_1 & \text{if } \sigma_1 \sigma_2 = -1. \end{cases}
$$

In the particular case of $\sigma_1 \sigma_2 = -1$,

$$\rho_1 \rho_2 < 1, \quad \text{if } S > 0, \\ \rho_1 \rho_2 > 1, \quad \text{if } S < 0.
$$

Moreover $\rho_1$ and $\rho_2$ must satisfy the algebraic constraint given by Eq. (43),

$$\rho_1^2 = \frac{(\sigma_1 l_1 - \sqrt{\Delta})^2 - l_2^2}{(\sigma_1 l_1 + \sqrt{\Delta})^2 - l_2^2},
$$

$$\rho_2^2 = \frac{(\sigma_2 l_2 - \sqrt{\Delta})^2 - l_1^2}{(\sigma_2 l_2 + \sqrt{\Delta})^2 - l_1^2},
$$

confirming that $l_1 = 0 \Leftrightarrow \rho_1 = 1$ and $l_2 = 0 \Leftrightarrow \rho_2 = 1$. The expressions for $\rho_1^2$ and $\rho_2^2$ must be positive, so intervals of allowed values for $l_1$ are implied. These intervals can be expressed in terms of $S$ and $I_4$, which themselves have allowed values to be determined by the properties of the Weierstrassian curves studied in Section 5. If $\sigma_1 \sigma_2 = 1$,

$$\frac{1}{2} (\sigma_2 S - \sqrt{S^2 + 4I_4}) < l_1 < \frac{1}{2} (\sigma_2 S + \sqrt{S^2 + 4I_4}).
$$

However, if $\sigma_1 = 1$ and $\sigma_2 = -1$,

$$0 \leq l_4 < \frac{1}{4} S^2,
$$

and either

$$l_1 < \frac{1}{2} (\sigma_1 S - \sqrt{S^2 - 4I_4}) \Leftrightarrow \rho_1 > \rho_2,
$$

or

$$l_1 > \frac{1}{2} (\sigma_1 S + \sqrt{S^2 - 4I_4}) \Leftrightarrow \rho_1 < \rho_2.
$$

Thence, the solutions for the auxiliary variables $\mu_j, j = 1, 2$, are obtained from substitution of Eqs. (55) and (56) into Eq. (42),

$$\mu_1 = -\frac{l_1}{2} - \frac{1}{2} \left(\frac{1 + \rho_1 e^{\rho_1} \Sigma}{1 - \rho_1 e^{\rho_1} \Sigma}\right) \frac{\sqrt{\Delta}}{\varphi(x + \omega') - \varphi(k)} - \frac{1}{6} \left(\frac{i \varphi'(x + \omega')}{\varphi(x + \omega') - \varphi(k)}\right).
$$

In the uncoupled case, $I_4 = 0$, the solutions for $\mu_1$ and $\mu_2$ in Eq. (65) become

$$\mu_1 = \mu_2 = -\frac{l_1}{2} - \frac{1}{2} \frac{S}{\varphi(x + \omega') - \varphi(k)} - \frac{1}{6} \left(\frac{i \varphi'(x + \omega')}{\varphi(x + \omega') - \varphi(k)}\right),
$$

the same solution obtained in [15]. Moreover, if we set $X = \varphi(x + \omega')$ and $Y = \varphi'(x + \omega')$, then the transformation to the projection of the point on the invariant spectral curve $\lambda(X, Y)$ given by Eq. (20) shows that when $I_4 = 0$,

$$\mu = \mu_1 = \mu_2 = \lambda(\varphi(x + \omega'), \varphi'(x + \omega')), \quad \text{if } \varphi(x + \omega') \neq \varphi(k),
$$

where $\mu$ is the closed orbit of the auxiliary variable when $I_4 = 0$, in agreement with [15]. Eq. (65) shows that, when $I_4 > 0$, the auxiliary variables $\mu_1$ and $\mu_2$ do not follow closed orbits because $\Sigma$ is only quasiperiodic. However the circle constraint (43) describes the bounded loci, assuming $l_1 l_2 \neq 0$, of $\mu_1$ and $\mu_2$ implicitly, when $I_4 > 0$.

$$I_4|\mu_1 - \mu|^2 + \sigma_2 (\mu_1 + \mu^*) - \mu - \mu^* I_4
$$

$$- \frac{l_2}{(\varphi(x + \omega') - \varphi(k))^2} I_4 = 0,
$$

$$I_4|\mu_2 - \mu|^2 + \sigma_2 (\mu_2 + \mu^*) - \mu - \mu^* I_4
$$

$$- \frac{l_1}{(\varphi(x + \omega') - \varphi(k))^2} I_4 = 0,
$$

where $\mu$ is the closed orbit defined in Eq. (67). Since $\mu = \lambda(\varphi(x + \omega'), \varphi'(x + \omega'))$, $\varphi(x + \omega')$ and $\varphi'(x + \omega')$ all have the same period, Eq. (68) shows that for each value of $x$, $\mu_1$ and $\mu_2$ each lie on a circle and return to these circles periodically in $x$.

4. Elliptic solutions

The solutions of the Manakov system of coupled nonlinear Schrödinger equations can now be obtained from the auxiliary variables via the equations

$$\frac{d\ln u}{dx} = 3i\mu_1 + 3if_1,
$$

$$\frac{d\ln v}{dx} = 3i\mu_2 + 3if_1.
$$

Using the fact that

$$\frac{d\ln S}{dx} = \frac{\varphi'(x)}{\varphi(x + \omega') - \varphi(k)},
$$

the integration can be performed to obtain

$$u = \gamma_1 \exp \left(\frac{3lf_1 x + if_1}{2} \left(1 - \rho_1 e^{\rho_1} \Sigma\right) \varphi(x + \omega') - \varphi(k)\right) \times \sqrt{\varphi(x + \omega') - \varphi(k)},
$$

$$v = \gamma_2 \exp \left(\frac{3lf_1 x + if_1}{2} \left(1 - \rho_2 e^{\rho_2} \Sigma\right) \varphi(x + \omega') - \varphi(k)\right) \times \sqrt{\varphi(x + \omega') - \varphi(k)},
$$
where $\gamma_1, \gamma_2 \in \mathbb{R}$ are constants of integration and $\phi_1, \phi_2 \in \mathbb{R}$ are arbitrary phase factors due to the symmetry of the original Manakov system. However $\gamma_1$ and $\gamma_2$ are not arbitrary but must satisfy Eqs. (36) and (38). If $\sigma_1\sigma_2 = 1$, then

$$
\gamma_1^2 - \frac{\rho_2}{(\rho_1 + \rho_2)(1 + \rho_1\rho_2)},
$$

$$
\gamma_2^2 = \frac{\rho_2}{(\rho_1 + \rho_2)(1 + \rho_1\rho_2)},
$$

while if $\sigma_1 = 1$ and $\sigma_2 = -1$, the quantity $|u|^2 - |v|^2$ does not change sign, so if $|u|^2 - |v|^2 > 0$, then

$$
\gamma_1^2 = \frac{\rho_2}{(\rho_1 + \rho_2)(1 - \rho_1\rho_2)},
$$

$$
\gamma_2^2 = \frac{\rho_2}{(\rho_1 + \rho_2)(1 - \rho_1\rho_2)},
$$

while if $|u|^2 - |v|^2 < 0$, then

$$
\gamma_1^2 = \frac{\rho_2}{(\rho_1^2 + \rho_2^2)(1 - \rho_1^2\rho_2^2)},
$$

$$
\gamma_2^2 = \frac{\rho_2^2}{(\rho_1^2 + \rho_2^2)(1 - \rho_1^2\rho_2^2)}.
$$

Together, Eqs. (73) and (74) imply further constraints on the constants of integration,

$$
(\rho_2 - \rho_1)(1 - \rho_1\rho_2) > 0, \quad \text{if } |u|^2 - |v|^2 > 0,
$$

$$
(\rho_2 - \rho_1)(1 - \rho_1\rho_2) < 0, \quad \text{if } |u|^2 - |v|^2 < 0.
$$

Using a standard addition theorem for the Weierstrass elliptic functions [9], the fact that $\sigma$ is an odd function and the quasi-periodicity of $\sigma$, the full solutions can be written in terms of the single-phase variable $\xi = x - 3ft + \xi_0$, where $\xi_0 \in \mathbb{R}$ is an arbitrary phase shift, as given in Box II.

Examining the solutions in Eq. (76) shows that although the locus of $\mu_1$ or $\mu_2$ is unbounded when either (i) $I_1 = 0, \rho_1 = 1$, or (ii) $I_2 = 0, \rho_2 = 1$, the solutions $u$ and $v$ themselves remain bounded and $u = 0$ or $v = 0$ at some value of the single-phase variable $\xi$. The formula in Eq. (76) for the solution of the coupled nonlinear Schrödinger equations (1) is the central result of this paper and is similar to the formula obtained in [19], except with a different parametrization. The current parametrization will now be used to give an effective characterization of the reality conditions and to study the limiting cases.

5. Reality conditions

For fixed $\sigma_1, \sigma_2 = \pm 1$ and phase-shift $\xi_0 \in \mathbb{R}$, $\theta_2$ is determined by $\theta_1$, so each solution in Eq. (76) is parametrized as a three-dimensional torus by the variables $(\phi_1, \phi_2, \theta_1) \in [0, 2\pi)^3$. This torus is defined by the values of six real parameters $(\alpha, f_1, \rho_1, \rho_2, g_2, g_3) \in \mathbb{R}^6$, however these six parameters are not arbitrary but must lie in a set of allowed values that satisfy all the reality conditions encountered in the construction of the bounded single-phase solution with real quasiperiod.

The parameters $g_2$ and $g_3$ define the allowed Weierstrassian curve and, hence, the functions $\wp, \zeta$ and $\sigma$ and the constant $\kappa$.

However $g_3 = 108 \left( \alpha - \frac{1}{4} \right)^3 - 3 \left( \alpha - \frac{1}{4} \right)^2 g_2 + 9\Delta,$

so the set of allowed values of $g_2$ and $g_3$ is determined by the set of allowed values of $\alpha, f_1, g_2$ and $\Delta$. The parameters $\rho_1$ and $\rho_2$ are determined by $I_1, I_2$ and $\Delta$. Thus all possible solutions are determined by the set of allowed values of $\alpha, f_1, I_1, I_2, g_2$ and $\Delta$.

Since $S = \sigma_1 I_1 + \sigma_2 I_2$, the parameter $I_2$ can be replaced by $S$ and hence $\Delta = S^2 + 4\rho_1\rho_2 I_4$ can be replaced by $I_4$. So it is sufficient to determine the set of allowed values of $(\alpha, f_1, S, g_2, I_1, I_4) \in \mathbb{R}^6$.

Now $g_3 = g_{30} + 36\sigma_1\sigma_2 I_4$,

so when $I_4 = 0$, the allowed cubic Weierstrassian curve $\tilde{C}_0$ is determined by $(\alpha, f_1, S, g_2) \in \mathbb{R}^4$ and the fact that increasing $I_1$ has the effect of translating the graph of the cubic curve $y = 4x^3 - g_2x - g_3$ vertically. Thus the allowed values of $I_1 > 0$ are determined by the set of allowed values of $(\alpha, f_1, S, g_2) \in \mathbb{R}^4$ and by the graph of the cubic curve $\tilde{C}_0$ and, in the case $\sigma_1\sigma_2 = -1$, by inequality (62), i.e., $\Delta > 0$. The allowed values of $I_4$ are subsequently determined by $S$ and $I_4$ using inequalities (61), (63) and (64). Finally the values of $(\alpha, f_1, S, g_2) \in \mathbb{R}^4$ are expressible in terms of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{C}^4$, the four zeros of $F_2$ defined in Eq. (25) with real coefficients. Therefore all bounded single-phase elliptic solutions $u$ and $v$ of the form (76) may be classified by the six-real-dimensional subspace of allowed values of the parameters within $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{C}^4$ and $(I_1, I_4) \in \mathbb{R}^2$.

5.1. Focusing–defocusing case

When $\sigma_1 = \sigma_2 = 1$, the coupling has two focusing terms and

$$
v - 3 \left( \alpha - \frac{1}{4} \right) - 3 |u|^2 - |v|^2 < 0,
$$

so bounded periodic solutions exist if and only if the three distinct zeros $e_1 < e_2 < e_3$ of $p(x) = 4x^3 - g_2x - g_3$ satisfy

$$
e_3 \leq v \leq e_2 < 3 \left( \alpha - \frac{1}{4} \right).
$$

The inflection point of the graph $y = p(x)$ is always at $x = 0$ and, since the roots are real and distinct, $g_2 > 0$. When $I_4 = 0$, the values of $e_1, e_2, e_3$ are given in terms of the four zeros of $F_4$, viz., $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$, by Eqs. (28) and (29). Eq. (78) shows that, since $\sigma_1\sigma_2 = 1$, the graph of $y = p(x)$ will be translated vertically downwards as $I_4 \geq 0$ increases. There are three cases to consider.

(i) Four distinct real zeros, $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$. Eq. (30) shows that $g_2 > 0$ and Eqs. (28) and (29) show that the three roots $e_1, e_2, e_3$ are real but that $3(\alpha - \frac{1}{4})^2 < e_2 < e_3 < e_1$, so the reality condition (80) is not satisfied for $I_4 = 0$. Moreover

$$
\frac{d^2}{dx^2} = \frac{d^2}{g_{30}^2} - \frac{36}{12e_1^2 - g_2} - \frac{36}{12e_2^2 - g_2} > 0,
$$

so the reality condition (80) is not satisfied for $I_4 \geq 0$. No periodic bounded solutions exist in this case.

(ii) Two distinct real zeros $\lambda_1 < \lambda_2$ and a complex–conjugate pair of zeros $\lambda_3 = c + id$ and $\lambda_4 = c - id$, with $d > 0$. In this case Eq. (30) becomes

$$
g_2 = \frac{27}{4}(\lambda_1 - c)(\lambda_2 + c + d^2) - \frac{81}{4}d^2(\lambda_1 - \lambda_2)^2.
$$

which may be positive or negative. The single real root $e_1$, given by Eq. (28) is greater than $3(\alpha - \frac{1}{4})^2$ and $e_2^2 = e_3$ , thus the reality condition (80) can never be satisfied. In particular, since the curve is translated downwards, an interval of oscillation of $v$ exists for $I_4 \geq 0$ if and only if $g_2 > 0$ and the height of the inflection point on the graph $y = p(x)$ for $I_4 = 0$ is positive, viz., $p(0) = -g_{30} > 0$. Now $g_{30} = 4e_1e_2e_3 = 4e_1^3|e_1|\epsilon_1, \text{ so } I_4 > 0 \iff e_1 < 0$. However Eq. (29) implies that, since $d > 0$,

$$
e_3 = \frac{3}{4}(\lambda_1 + \lambda_2 + c)^2 + \frac{15}{4}d^2 + \frac{3}{2}d^2 > 0.
$$

Therefore, there are no bounded periodic solutions in this case.
\[ u(t, x) = r_1 \left( \frac{\exp(-\zeta(\kappa)\xi - \eta'')\sigma(\xi + \omega' + \kappa) - \rho_1 e^{\theta_1} \exp(\zeta(\kappa)\xi + \eta'')\sigma(\xi + \omega' - \kappa)}{\sigma(\xi + \omega')\sigma(\kappa)} \right), \]  
\[ v(t, x) = r_2 \left( \frac{\exp(-\zeta(\kappa)\xi - \eta'')\sigma(\xi + \omega' + \kappa) - \rho_2 e^{\theta_2} \exp(\zeta(\kappa)\xi + \eta'')\sigma(\xi + \omega' - \kappa)}{\sigma(\xi + \omega')\sigma(\kappa)} \right), \]  
where, for \( j = 1, 2, \)
\[ r_j = \gamma_j \exp \left( 9i\omega t + \frac{3i}{2} f_1 \xi + i\phi_j \right). \]  

\section*{Box II.}

(iii) Two distinct complex-conjugate pairs of non-real zeros \( \lambda_1 = a + ib, \lambda_2 = a - ib, \lambda_3 = c + id, \lambda_4 = c - id, \) with \( b, d > 0. \) In this case Eq. (30) implies
\[ g_2 = 81b^2d^2 + \frac{27}{4}((a - c)^2 + b^2 + d^2)^2 > 0, \]  
and the three real roots of \( p(x) = 0 \) are given by Eq. (28), where \( e_2 \) and \( e_3 \) need to be renamed so that \( e_3 < e_2. \)
\[ e_1 = \frac{9}{4}(a - c)^2 + 3 \left( \alpha - \frac{1}{4}f_1^2 \right), \]
\[ e_2 = \frac{-9}{4}(b - d)^2 + 3 \left( \alpha - \frac{1}{4}f_1^2 \right), \]
\[ e_3 = \frac{-9}{4}(b + d)^2 + 3 \left( \alpha - \frac{1}{4}f_1^2 \right). \]

Both \( e_2 \) and \( e_3 \) are less than \( 3(\alpha - \frac{1}{4}f_1^2), \) producing an interval of oscillation for \( \nu \) that satisfies the reality condition (80). The derivative of the larger root \( e_2 \) with respect to \( l_4 \) satisfies an inequality similar to (81).
\[ \frac{de_2}{dl_4} = \frac{de_1}{dl_4} - g_2 = \frac{36}{12e_2^1 - g_2} = \frac{36}{p(e_2)} < 0, \]

implying that \( e_2 \) will decrease for increasing \( l_4 \geq 0. \) Therefore the reality condition (80) remains satisfied for \( l_4 \geq 0 \) in the interval for which \( p(x) \) continues to have three distinct real roots, viz.,
\[ 0 \leq l_4 < \frac{1}{36} \left( \frac{3\sqrt{3} g_{30}^{3/2} - g_{30}}{2} \right). \]  

Thus any two complex-conjugate pairs for \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are allowed, these four real parameters determine all possible allowed values of \( \alpha, f_1, g_2, g_{30} \in \mathbb{R}, \) from which \( g_{30} \in \mathbb{R} \) is also calculated using Eq. (23), \( l_4 \in \mathbb{R} \) is any real number in the interval given by (87), determined by \( g_2 \) and \( g_{30}. \) Finally \( l_1 \in \mathbb{R} \) is any number in the interval given by inequality (61).
\[ \frac{1}{2}(S - \sqrt{S^2 + 4l_4}) < l_1 < \frac{1}{2}(S + \sqrt{S^2 + 4l_4}). \]

5.2. Defocusing–defocusing case

When \( \sigma_1 = \sigma_2 = -1, \) the coupling has two defocusing terms and
\[ v - 3 \left( \alpha - \frac{1}{4}f_1^2 \right) = |u|^2 + |v|^2 > 0, \]

so bounded periodic solutions exist if and only if the three distinct zeros \( e_3 < e_2 < e_1 \) of \( p(x) = 4x^2 - g_2x - g_3 \) satisfy
\[ 3 \left( \alpha - \frac{1}{4}f_1^2 \right) < e_3 \leq v \leq e_2. \]

The inflection point of the graph \( y = p(x) \) is always at \( x = 0 \) and, since the roots are real and distinct, \( g_2 > 0. \) When \( l_4 = 0, \) the values of \( e_1, e_2, \) and \( e_3 \) are given in terms of the four zeros of \( F_4 \) by Eqs. (28) and (29). Eq. (78) shows that, since \( \sigma_1\sigma_2 = 1, \) the graph of \( y = p(x) \) will be translated vertically downwards as \( l_4 \geq 0 \) increases. There are three cases to consider.

(i) Four distinct real zeros, \( \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4. \) Eq. (30) shows that \( g_2 > 0 \) and Eqs. (28) and (29) show that the three roots \( e_1, e_2, e_3, \) are real and that \( 3(\alpha - \frac{1}{4}f_1^2) < e_3 < e_2 < e_1, \) so that the reality condition (90) is satisfied for \( l_4 = 0. \) As in the focusing–focusing case, Eq. (81) shows that \( e_1 \) will increase as \( l_4 \geq 0 \) increases, so that the reality condition (90) will continue to be satisfied as long as the cubic equation \( p(x) = 0 \) has three real roots. Thus the allowed interval of \( l_4 \) is the same as in case (iii) for the focusing–focusing equation, viz.,
\[ 0 \leq l_4 < \frac{1}{36} \left( \frac{1}{3} \sqrt[3]{3} g_{30}^{3/2} - g_{30} \right). \]

To summarize, any four distinct values \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are allowed, these four real parameters determine all possible allowed values of \( \alpha, f_1, g_2, g_{30} \in \mathbb{R}, \) from which \( g_{30} \in \mathbb{R} \) is also calculated using Eq. (23), \( l_4 \in \mathbb{R} \) is any real number in the interval given by (91), determined by \( g_2 \) and \( g_{30}. \) Finally \( l_1 \in \mathbb{R} \) is any number in the interval given by inequality (61).
\[ \frac{1}{2}(S - \sqrt{S^2 + 4l_4}) < l_1 < \frac{1}{2}(S + \sqrt{S^2 + 4l_4}). \]

(ii) Two distinct real zeros, \( \lambda_1 < \lambda_2, \) and a complex-conjugate pair of zeros, \( \lambda_3 = c + id \) and \( \lambda_4 = c - id, \) with \( d > 0. \) The same reasoning as in the focusing–focusing case, Section 5.1 case (ii), shows that no bounded periodic solution can arise in this case.

(iii) Two distinct complex-conjugate pairs of zeros \( \lambda_1 = a + ib, \lambda_2 = a - ib, \lambda_3 = c + id, \) and \( \lambda_4 = c - id, \) with \( c, d > 0. \) In this case, \( g_2 > 0, \) from Eq. (84), and \( e_1, e_2, e_3 \in \mathbb{R} \) are still given by Eq. (85) that appeared in the third case of the focusing–focusing equations. Eq. (85) implies that \( e_3 < e_2 < 3(\alpha - \frac{1}{4}f_1^2), \) contradicting the reality condition (90). As \( l_4 \geq 0 \) increases, \( e_2 \) continues to be less than the constant value of \( 3(\alpha - \frac{1}{4}f_1^2) \) because of Eq. (86). Therefore the reality condition (90) is never satisfied and no bounded periodic solutions exist in this case.

5.3. Focusing–defocusing case

When \( \sigma_1 = -\sigma_2 = 1, \) the coupling is between a focusing term and a defocusing term. In this case, Eq. (40) gives
\[ |u|^2 - |v|^2 = 3 \left( \alpha - \frac{1}{4}f_1^2 \right) - \rho(x + \omega'), \]

but, as discussed earlier, for bounded periodic solutions with \( l_4 > 0, \) \( \Delta > 0 \) and \( |u|^2 \neq |v|^2, \) so \( |u|^2 - |v|^2 \) cannot change sign.
Consequently, since any allowed interval of oscillation of \( \varphi(x + \omega') \) must include the critical number of the relative maximum of 
\[
p(x) = 4x^3 - g_2x - g_3, \text{ viz., } x = -\sqrt{\frac{g_2}{12}},
\]
the sign of \(|u|^2 - |v|^2|\) is always the same as the sign of 
\[
3 \left( \alpha - \frac{1}{4}f_1^2 \right) + \sqrt{\frac{g_2}{12}}.
\]
The interval of oscillation of \(|u|^2 - |v|^2|\) changes continuously with \(I_4\) if \(I_4 = 0\) is an allowed value of \(I_4\), which is only true in cases (i) and (iii) below, consider the limit \(I_4 \to 0\) in Eqs. (13), (14) and (16). If \(I_1 \neq 0\) and \(I_2 \neq 0\) then \(|u| \neq 0\) and \(|v| \neq 0\), so \(\mu_2 \to \mu_1\) and 
\[
\frac{I_1}{I_2} \to \frac{|u|^2}{|v|^2},
\]
so that \(S = I_1 - I_2\) and \(|u|^2 - |v|^2|\) have the same sign for sufficiently small \(I_4\). However, since \(|u|^2 - |v|^2|\) cannot change sign, \(|u|^2 - |v|^2|\) will also have the same sign as \(S\) for all allowed values of \(I_4\), provided \(I_4 = 0\) is an allowed value.

As for the focusing–defocusing and defocusing–defocusing equations, when \(I_4 = 0\), the values of \(e_1\), \(e_2\) and \(e_3\) are given by Eqs. (28) and (29) in terms of the four zeros of \(F_0\), viz. \(\lambda_1, \lambda_2, \lambda_3, \lambda_4\). Unlike the focusing–defocusing and defocusing–defocusing equations, \(\sigma_0e_2 = 1\), so increasing \(I_4\) translates the graph of the cubic curve \(y = p(x)\) vertically upwards. There are again three cases to consider.

(i) Four distinct real zeros, \(\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4\). Eq. (30) implies that 
\[
g_2 > 0 \quad \text{and Eqs. (28) and (29) show that the three roots} \quad e_1, e_2, e_3, \quad \text{are real and that} \quad 3(\alpha - \frac{1}{4}f_1^2) < e_3 < e_2 < e_1.
\]
Bounded periodic solutions exist for \(I_4 \geq 0\) as long as (a) the three real roots remain distinct, (b) \(3(\alpha - \frac{1}{4}f_1^2)\) is less than the least root of \(p(x) = 0\), and (c) constraint (62) is satisfied. In particular, \(\Delta > 0\) if and only if 
\[
p \left( 3 \left( \alpha - \frac{1}{4}f_1^2 \right) \right) = -9(S^2 - 4I_4) = -9\Delta < 0.
\]
So conditions (b) and (c) are actually equivalent. Also condition (a) is equivalent to the local minimum of \(p(x)\) remaining negative,
\[
p \left( \frac{g_2}{\sqrt{12}} \right) = p \left( -\frac{g_2}{\sqrt{3}} \right) = 36I_4 - \left( \frac{g_2^{3/2}}{3\sqrt{3}} + g_{30} \right) < 0.
\]
Thus the interval of allowed \(I_4\) depends on the value of \(3(\alpha - \frac{1}{4}f_1^2)\) relative to \(-\sqrt{\frac{g_2}{12}}\) and is given by the intersection of the two intervals determined by Eqs. (95) and (96).

(a) If \(3(\alpha - \frac{1}{4}f_1^2) \leq -\sqrt{\frac{g_2}{12}}\), then 
\[
0 \leq I_4 < \frac{1}{36} \left( \frac{g_2^{3/2}}{3\sqrt{3}} + g_{30} \right) \leq \frac{S^2}{4},
\]
(b) if \(-\sqrt{\frac{g_2}{12}} < 3(\alpha - \frac{1}{4}f_1^2) < e_3\), then 
\[
0 \leq I_4 < \frac{S^2}{4} < \frac{1}{36} \left( \frac{g_2^{3/2}}{3\sqrt{3}} + g_{30} \right).
\]
Since \(v = \varphi(x + \omega')\) oscillates in the interval \([e_3, e_2]\), 
\[|u|^2 - |v|^2| = 3 \left( \alpha - \frac{1}{4}f_1^2 \right) - \varphi(x + \omega') < 0.\]
Now \(I_4 = 0\) is always an allowed value, so \(S < 0\) also. Eq. (74) gives the correct formulae for \(\gamma_1\) and \(\gamma_2\) and Eq. (75) shows that \((\rho_1 - \rho_1)(1 - \rho_1\rho_2) < 0\). Thus, by Eq. (59), since \(S < 0\), \(\rho_1\rho_2 > 1\). Therefore \(\rho_2 > \rho_1\) and Eq. (64) gives the allowed interval of \(I_1\) values.

(ii) Two distinct real zeros, \(\lambda_1 < \lambda_2\), and a complex-conjugate pair of zeros, \(\lambda_3 = c + id\) and \(\lambda_4 = c - id\) with \(d > 0\). In this case \(p(x) = 0\) has only one real root, so \(I_4 = 0\) is not an allowed value. In fact \(g_2\) may be positive or negative according to Eq. (82), so that a necessary condition for a bounded interval of oscillation between \(e_1\) and \(e_2\) for \(I_4 > 0\) is that \(g_2 > 0\). Assuming that 
\[
g_2 = \frac{27}{4}((\lambda_1 - c)(\lambda_2 - c) + d^2)^2
\] 
\[
- \frac{81}{4}d^2(\lambda_1 - \lambda_2)^2 > 0,
\]
there will be a bounded periodic solution for \(I_4\) in the interval for which \(p(x)\) has three real zeros, provided that \(|u|^2 - |v|^2|\) cannot change sign on that interval. Eq. (83) shows that the single real root of \(p(x) = 0\) is positive and thence the inflection point on the graph of \(y = p(x)\) is below the \(x\)-axis when \(I_4 = 0\). Depending on the value of \(3(\alpha - \frac{1}{4}f_1^2)\), it is possible for \(|u|^2 - |v|^2|\) to be positive or negative but it cannot change sign on the interval of oscillation. Since the local maximum of \(p(x)\) occurs at \(x = -\sqrt{\frac{g_2}{12}}\) and an interval of oscillation occurs if and only if this maximum is positive,
\[ 0 < -\frac{1}{36} \left( \frac{g_{20}^{3/2}}{\sqrt{3}} + g_{30} \right) < \frac{S^2}{4}. \]  
(105)

Assuming \( g_{20} > 0 \) and inequality (104), the inequalities (100), (101) and (103) simultaneously determine the allowed interval of \( I_1 \) which depends on the value of \( 3(\alpha - \frac{1}{4}t^2) \), using inequalities (104) and (105), as follows.

(a) If \( 3(\alpha - \frac{1}{4}t^2) \leq -\frac{\sqrt{2}}{3} \), then

\[ 0 < -\frac{1}{36} \left( \frac{g_{20}^{3/2}}{\sqrt{3}} + g_{30} \right) < I_4 \]

\[ < \frac{1}{36} \left( \frac{g_{20}^{3/2}}{\sqrt{3}} + g_{30} \right) \leq \frac{S^2}{4}, \]  
(106)

(b) if \( \sqrt{\frac{4}{3}} < 3(\alpha - \frac{1}{4}t^2) < -\sqrt{\frac{4}{3}} \) or \( -\sqrt{\frac{4}{3}} < 3(\alpha - \frac{1}{4}t^2) < \frac{\sqrt{2}}{3} \), then

\[ 0 < -\frac{1}{36} \left( \frac{g_{20}^{3/2}}{\sqrt{3}} + g_{30} \right) < I_4 < \frac{S^2}{4} \]

\[ < \frac{1}{36} \left( \frac{g_{20}^{3/2}}{\sqrt{3}} + g_{30} \right). \]  
(107)

Thus case (ii) can be summarized as follows:

(a) If \( 3(\alpha - \frac{1}{4}t^2) + \sqrt{\frac{4}{3}} > 0 \), then \( |u|^2 - |v|^2 > 0 \) and \( \gamma_1 \) and \( \gamma_2 \) are given by Eq. (73). Eq. (75) implies that \( \rho_1 - \rho_1(1 - \rho_1 \rho_2) > 0 \). If \( S > 0 \) also, then \( \rho_1 \rho_2 < 1 \) by Eq. (59) and so \( \rho_1 < \rho_2 \) which implies that the interval of allowed values of \( I_1 \) is given by Eq. (64), in agreement with case (ii) below. However, if \( S < 0 \), then \( \rho_1 \rho_2 > 1 \) and \( \rho_1 > \rho_2 \) and the interval of allowed values of \( I_1 \) is given by Eq. (65), in agreement with case (i) above.

(iii) Two distinct complex-conjugate pairs of zeros \( \lambda_1 = a + ib, \lambda_2 = a - ib, \lambda_3 = c + id, \lambda_4 = c - id, \) with \( b, d > 0 \). In this case, Eq. (84) shows that \( g_{20} > 0 \) and Eq. (85) implies that \( p \) has three real zeros when \( I_4 = 0 \). The interval of allowed values of \( I_4 \) is

\[ 0 \leq I_4 < \frac{S^2}{4} \leq \frac{1}{36} \left( \frac{g_{20}^{3/2}}{\sqrt{3}} + g_{30} \right). \]  
(108)

with the second equality if and only if

\[ 3 \left( \alpha - \frac{1}{4}t^2 \right) = \frac{\sqrt{2}}{12}. \]

Now \( e_3 < e_2 < 3(\alpha - \frac{1}{4}t^2) < e_1 \), so

\[ |u|^2 - |v|^2 = 3 \left( \alpha - \frac{1}{4}t^2 \right) - \frac{\sqrt{3}}{3} \times \frac{\gamma}{(x + \omega^2)} > 0 \]

and since \( I_4 = 0 \) is an allowed value, both \( |u|^2 - |v|^2 > 0 \) and \( S > 0 \). Thus the values of \( \gamma_1 \) and \( \gamma_2 \) are given by Eq. (73) and the allowed interval of \( I_1 \) is given by Eq. (64).

6. Limiting cases

There are two limiting cases of interest, the soliton limit of infinite real period where the elliptic modulus \( k \to 1 \), viz.,

\[ e_2 \to e_1, \]  
and the small-wave-modulation limit where the elliptic modulus \( k \to 0 \), viz.,

\[ e_3 \to e_2, \]  
and the signed sum of the squared amplitudes of the two waves is given by Eq. (40). In the soliton limit, \( k \to 1 \),

\[ \sigma_1 |u|^2 + \sigma_2 |v|^2 = 3 \left( \alpha - \frac{1}{4}t^2 \right) \left( \sqrt{\frac{g_{20}}{12}} + \sqrt{\frac{3g_{20}}{4}} \right) \]

\[ \times \left( \frac{3g_{20}}{4} \right) \left( x - 3ft + 3t + \xi_0 \right). \]  
(109)

In the small-wave-modulation limit, \( 0 < k^2 \ll 1 \),

\[ \sigma_1 |u|^2 + \sigma_2 |v|^2 \approx \sqrt{\frac{g_{20}}{12}} + 3 \left( \alpha - \frac{1}{4}t^2 \right) \]

\[ + \frac{3g_{20}}{16} k^2 \cos((12g_{20})^2 (x - 3ft + 3t + \xi_0)). \]  
(110)

Note that Eq. (36) implies that the leading term in the above small-wave-modulation limit expansion is necessarily the same sign as the sum \( \sigma_1 |u|^2 + \sigma_2 |v|^2 \), since whenever the reality conditions are satisfied the interval of oscillation of \( v \) will contain \( v = -\sqrt{\frac{g_{20}}{12}} \), the critical number of \( p(x) \) at its local maximum.

6.1. Soliton limit

6.1.1. Focusing–focusing case

It was shown previously that the reality condition for this case requires two distinct complex-conjugate pairs \( \lambda_1 = a + ib, \lambda_2 = a - ib, \lambda_3 = c + id, \lambda_4 = c - id, \) with \( b, d > 0 \). Eqs. (30) and (85) give explicit expressions for \( g_{20} \) and \( e_1, e_2, e_3 \in \mathbb{R} \), when \( I_4 = 0 \). The interval of allowed \( I_4 \) is given by Eq. (87).

Since the graph of \( y = p(x) = 4(x - e_1)(x - e_2)(x - e_3) \) is translated vertically downwards for increasing \( I_4 > 0 \), the soliton limit \( e_2 = e_1 \) occurs only when \( I_4 = 0 \). Therefore Eq. (85) shows that the soliton limit is \( I_4 = 0, a = c = b = d \). Moreover, the repeated root of \( p(x) = 0 \) is \( x = \sqrt{\frac{g_{20}}{12}} \), so Eq. (85) implies

\[ e_1 = e_2 = e_3 = \sqrt{\frac{g_{20}}{12}} = 3 \left( \alpha - \frac{1}{4}t^2 \right). \]  
(111)

Thus Eq. (109) shows that the sum of the amplitudes produces the well-known bright soliton,

\[ |u|^2 + |v|^2 = 9b^2 \sinh^2 \left( 3b \left( x - 64a + \xi_0 \right) \right). \]  
(112)

Care must be taken to show that this limit exists in a non-trivial sense from within the collection of allowed elliptic solutions. For fixed \( c, d \in \mathbb{R}, d > 0 \), both \( a, b \in \mathbb{R}, b > 0 \), can be chosen freely to make \( \delta = \sqrt{(a - c)^2 + (b - d)^2} \) arbitrarily small but non-zero. It is also assumed that \( \delta \) is bounded so that \( a \) and \( b \) are bounded and \( \delta \) is sufficiently small so that \( b \) is bounded away from zero. However \( I_4 \) and \( I_4 \) must lie in the intervals given by Eqs. (87) and (88), respectively. In particular,

\[ g_{20}^2 - 27g_{30}^2 = \frac{531441}{16} b^2 d^2 ((b + d)^2 + (a - c)^2) \delta^4 > 0, \]  
(113)

so that

\[ \frac{1}{36} \left( \frac{1}{3\sqrt{3}} g_{20}^{3/2} - g_{30} \right) > 0 \]  
(114)

and the vanishing interval of allowed values for \( I_4 \),

\[ 0 \leq I_4 < \frac{1}{36} \left( \frac{1}{3\sqrt{3}} g_{20}^{3/2} - g_{30} \right) = O(\delta), \]  
(115)
always contains non-zero \( l_a \), even for arbitrarily small but non-zero \( \delta \). Similarly,

\[
S = \frac{9}{4} (c - a)(b^2 - d^2) \neq 0 \tag{116}
\]

when \( \delta > 0 \), so the interval of allowed values of \( l_1 \) given by (88) always contains the non-trivial interval \( 0 \leq l_1 < S \) or \( S < l_1 \leq 0 \), depending on whether \( S > 0 \) or \( S < 0 \), respectively. Thus the soliton limit for \( u \) and \( v \) is the limit of non-trivial members of the class of elliptic solutions given by (76) and can be obtained in steps, first \( l_4 \to 0 \), then \( a \to c \) and finally, \( b \to d \). This extends the procedure used in the scalar equation case [9] to the coupled system.

For fixed \( a \neq c \) and \( b \neq d \), \( \alpha = ac + \frac{1}{2}b^2 + \frac{1}{2}d^2, f_1 = -a - c \) and \( S \) is given by Eq. (116). Assuming \( S > 0 \), if \( l_4 \to 0 \), then

\[
\begin{align*}
\rho_1^2 &\to \frac{2l_1}{2S l_1} I_1 + O(l_4^2), \\
\rho_2^2 &\to \frac{2l_2}{2S l_2} I_2 + O(l_4^2), \\
\gamma_1^2 &\to \frac{l_1}{l_1 + l_2} + O(l_4), \\
\gamma_2^2 &\to \frac{l_2}{l_1 + l_2} + O(l_4),
\end{align*}
\tag{117}
\]

for any \( l_1 \in (0, S) \). Also \( l_2 = S - l_1 \) for each allowed \( l_1 \). Thus, in the limit \( l_4 \to 0 \), the solution given in Eq. (76) becomes

\[
\begin{align*}
u(t, x) &= \sqrt{\frac{l_1}{l_1 + l_2}} e^{\theta(x) + \frac{2}{3}f_1 t + \frac{2}{3} \phi_1} 	imes \exp(-\zeta(x) \xi - (\eta - \eta') \sigma(\xi + \omega + \kappa)) \\
&\times \frac{\sigma(\xi + \omega + \kappa)}{\sigma(\xi + \omega') \sigma(\kappa)}.
\end{align*}
\tag{118}
\]

Note that the limiting expressions for \( u \) and \( v \) given in (118) can be shown to be valid for \( l_4 \to 0 \) and \( l_1 \to S \) as well.

As \( c \to a, S \to 0 \) and the interval of allowed values \( l_1 \in [0, S] \) for the limiting expressions (118) collapses to \( l_1 \to 0 \). However, \( S \neq 0 \) for each \( c \neq a \) and \( b \neq d \), so it is possible to choose \( l_1 = \gamma_1^2 S \) and \( l_2 = \gamma_2^2 S \) where \( \gamma_1^2 \in [0, 1] \) is arbitrary but fixed and \( \gamma_2^2 = 1 - \gamma_1^2 \), so that \( l_1 \) is in the allowed interval for each \( S > 0 \), even as \( S \) itself is decreasing to zero. In the limit \( c = a, \alpha = a^2 + \frac{1}{2}b^2 + \frac{1}{2}d^2, f_1 = -2a \) and \( e_1 = 3(\alpha - b^2) = \varphi(\kappa) \), from Eq. (50), so \( \kappa = \omega \) (or a congruent point) and \( \zeta(\kappa) = \zeta(\omega) = \eta \).

Using the identities in Appendix, the limit of the expression for \( u(t, x) \) in (118) as \( c \to a \) is

\[
\begin{align*}
u(t, x) &= e^{\theta + \frac{2}{3}f_1 t + \frac{2}{3} \phi_1} e^{-\varphi - \varphi'} \frac{\sigma(\xi + \omega + \omega)}{\sigma(\xi + \omega')} \\
&\times \frac{\sigma(\xi + \omega + \omega)}{\sigma(\xi + \omega') \sigma(\kappa)}
\end{align*}
\tag{119}
\]

where the elliptic modulus is

\[
k = \sqrt{\frac{e_3 - e_2}{e_1 - e_3}} = \frac{4bd}{(b + d)^2}. \tag{120}
\]

The limit as \( c \to a \) for \( v(t, x) \) is the same as for \( u(t, x) \), except \( \gamma_1 \) and \( \phi_1 \) are replaced by \( \gamma_2 \) and \( \phi_2 \).

Finally, the soliton limit \( k = 1 \) is obtained by setting \( b = d \), so

\[
\alpha = a^2 + b^2, f_1 = -2a \text{ and}
\]

\[
u(t, x) = 3b \gamma_1 e^{\phi_1} \text{sech}(3b(x - 6at + \xi_0)) e^{-3a(x + 9b^2 - d^2)t - 3a\xi_0}.
\tag{121}
\]

The reason the Manakov soliton is recovered in the focusing–focusing soliton limit in Eq. (121) but not in (20) can be explained in terms of the deck transformations of the branched covering of the Riemann sphere associated with the trigonometric curve (11). The deck transformations studied in [20] do not correspond to spectral curves satisfying the reality conditions found in Section 5.1 in a region of the parameter space containing the soliton limit.

The focusing–focusing soliton limit at \( l_4 = 0 \), \( a = c, b = d \), was obtained by first letting \( l_4 \to 0^+ \) while \( a \neq c \) and \( b \neq d \). For fixed \( c \neq a \), \( b \neq d \), the branch points of the spectral curve are continuous functions of \( l_4 \) given by the six roots of the discriminant of the curve, viz., \( D = 4A(\lambda)^3 + 27B(\lambda)^2 = 0 \). The branch points are distinct unless the discriminant of \( D \) itself is zero. The discriminant of \( D \) is proportional to the product

\[
\Delta_4 \left( I_4 - \frac{1}{36} \left( 1 - \frac{1}{3}\sqrt{3} \frac{g_3^{1/2}}{g_0} \right) \right)
\times \left( I_4 - \frac{1}{36} \left( 1 - \frac{1}{3}\sqrt{3} \frac{g_3^{1/2}}{g_0} \right) \right) P(\varepsilon)^3
\tag{123}
\]

where \( A = S^2 + 4l_4 > 0 \) and \( P \) is a polynomial in \( \varepsilon = b^2 - d^2 \). For fixed \( a, c, d \in \mathbb{R} \), with \( d > 0, a \neq c \), there exists a sufficiently small neighborhood \( \varepsilon_0 \) of zero, such that if \( \varepsilon \in \varepsilon_0 \), then (i) \( b \) is bounded and bounded away from zero, (ii) the interval (87) of allowed \( l_4 \) is
non-trivial and (iii) for all such \( b \) and \( I_4 \) in this neighborhood of the soliton limit,

\[
P(e) = (64I_4^2 + 243d^6(a - c)^6) + O(e)
\geq 243^2d^{12}(a - c)^{12} + O(e) > 0.
\]  

(124)

In this sufficiently small neighborhood of the soliton limit, there are only three real \( I_4 \) roots of \( D \), viz.,

\[
I_4 = 0,
\]

\[
I_4 = \frac{1}{36} \left( \frac{1}{3} \sqrt[3]{g_2^2/3 - g_30} \right) > 0,
\]

\[
I_4 = \frac{1}{36} \left( -\frac{1}{3} \sqrt[3]{g_2^2/3 - g_30} \right) < 0,
\]

(125)

where the inequalities follow from Eq. (113). Thus, in this neighborhood of the soliton limit, the branch points of the discriminant must be distinct for all \( I_4 \) satisfying the reality condition of Eq. (87), except when \( I_4 = 0 \). When \( I_4 = 0 \), the repeated root of \( D \) is given by the root of \( F_1 = 0 \) and the other four roots by the four roots of \( F_3 = 0 \), as defined in (25). Since the leading coefficient of \( F_1 \) is \( S \), if \( a = c \) or \( b = d \), the root of \( F_1 = 0 \) actually moves off to the point at infinity, but when \( a \neq c \) and \( b \neq d \), this repeated root of the discriminant is finite.

Now consider the deck transformations of the sheets of the branched coverings. For the purposes of this discussion, the deck transformations above the six simple branch points can be represented, modulo automorphisms of \( S_3 \), the permutation group on three elements, as an unordered set of six transpositions. If the three sheets are labeled 1, 2, and 3, then there is a single transposition of the elements 1, 2, and 3 for each of the simple branch points. However, for fixed \( d > 0 \), \( a \neq c \in \mathcal{S} \) and \( I_4 \) satisfying the reality condition (87), the branch points do not meet except when \( I_4 = 0 \). The spectral curve varies continuously with the \( I_4 \), so the deck transformations cannot change except when the branch points meet, which is only at \( I_4 = 0 \) in this neighborhood of the soliton limit. When \( I_4 = 0 \), the spectral curve factors and the quadratic factor must have a branched covering with four branch points connecting two sheets; the only possible deck transformations for the quadratic factor are \((1, 2), (1, 2), (1, 2), (1, 2))\}. Therefore, the deck transformations of the three-sheeted branch covering in this neighborhood of the soliton limit must be \((1, 2), (1, 2), (1, 2), (1, 2), (2, 3), (2, 3))\}, since these are the only deck transformations possible for an irreducible trigonal curve with the correct limit for the quadratic factor when \( I_4 = 0 \). In [20] the Manakov soliton was not recovered in the soliton limit because the only deck transformations studied were \((1, 2), (1, 2), (2, 3), (2, 3), (3, 1), (3, 1))\}, instead of the deck transformations \((1, 2), (1, 2), (1, 2), (1, 2), (2, 3), (2, 3))\} which correspond to solutions satisfying the reality conditions near the focusing–focusing soliton limit. The presence of two classes of deck transformations within the space of \( \mathcal{N} \)-phase solutions of a coupled nonlinear Schrödinger system is also discussed in [23,24].

### 6.1.3. Defocusing–defocusing case

It was shown in Section 5.2 that the reality condition for this case requires \( \lambda_1 \leq \lambda_2 < \lambda_3 < \lambda_4 \in \mathbb{R} \). Eqs. (28)–(30) give explicit expressions for \( g_2 \) and \( e_1, e_2, e_3 \in \mathbb{R} \), when \( I_4 = 0 \). The interval of allowed \( I_4 \) is given by Eq. (91). Since the graph of \( y = p(x) = 4(\sqrt{x} - e_3)(\sqrt{x} - e_3) \) is translated vertically downwards for increasing \( I_4 \) > 0, the soliton limit \( e_2 = e_1 \) occurs only when \( I_4 = 0 \). Eq. (28) shows that \( e_2 = e_1 \) implies that \( \lambda_2 = \lambda_3 \).

Eq. (109) shows that the sum of the amplitudes produces the well-known dark soliton against a constant positive background,

\[
|u|^2 + |v|^2 = \frac{9}{16}(\lambda_4 - \lambda_1)^2 - \frac{9}{4}(\lambda_4 - \lambda_2) \\
	imes (\lambda_2 - \lambda_1) \operatorname{sech}^2 \chi(t,x),
\]  

(126)

where

\[
\chi(t,x) = \frac{3}{2} \sqrt{(\lambda_4 - \lambda_2)(\lambda_2 - \lambda_1)} \\
	imes \left( x + \frac{3}{2} \lambda_1 + 2 \lambda_2 + \lambda_4 t + \xi_0 \right).
\]  

(127)

The soliton limit of \( u \) and \( v \) can be obtained in two steps. The first step is to calculate the limit \( I_4 \to 0 \), then \( \lambda_2 \to \lambda_3 \). In a process similar to the focusing–focusing case, we can assume without loss of generality that \( S > 0 \) and Eqs. (76) and (117) imply the limit \( I_4 \to 0 \) is

\[
u(t,x) = \frac{l_2}{l_1 + l_2} e^{\beta_1 \xi_1 + \beta_1 \xi_1} e^{-\beta_1 \xi_1 - \beta_1 \xi_1} \sigma(\xi + \omega \xi + \omega) \\
\frac{\sigma(\xi + \omega \xi + \omega)}{\sigma(\xi + \omega \xi + \omega)} \sigma(\omega),
\]  

(128)

\[
u(t,x) = \frac{l_2}{l_1 + l_2} e^{\beta_1 \xi_1 + \beta_1 \xi_1} e^{-\beta_1 \xi_1 - \beta_1 \xi_1} \sigma(\xi + \omega \xi + \omega) \\
\frac{\sigma(\xi + \omega \xi + \omega)}{\sigma(\xi + \omega \xi + \omega)} \sigma(\omega),
\]  

(129)

where \( \lambda_2 \to \lambda_3 \), the real period \( \omega \to \infty \) and, writing \( \beta = \sqrt{\frac{I_4}{2}} \), the limit of the complex period is

\[
\omega' = \frac{\pi r}{2 \sqrt{\beta}}.
\]  

Also \( \eta' = \zeta(\omega') = -\beta \omega' \) and the Weierstrassian functions have limits [9] given by

\[
\varphi(z) = \beta + 3 \beta \operatorname{csch}^2(\sqrt{3} \beta z),
\]

\[
\zeta(z) = -\beta z + \sqrt{3} \beta \coth(\sqrt{3} \beta z),
\]

\[
\sigma(z) = f \sqrt{3} \beta \sinh(\sqrt{3} \beta z) \exp \left( -\frac{1}{2} \beta z \right).
\]  

(130)

Thus

\[
\varphi(x) = 3 \left( x - \frac{1}{4} f^2 \right) = \beta + 3 \beta \operatorname{csch}^2(\sqrt{3} \beta k),
\]

implies

\[
\operatorname{csch}^2(\sqrt{3} \beta k) = \frac{-(\lambda_4 - \lambda_2)^2}{(\lambda_4 - \lambda_2)(\lambda_2 - \lambda_1)}.
\]  

(131)

\[
\operatorname{coth}^2(\sqrt{3} \beta k) = \frac{-(\lambda_4 - 2 \lambda_2 + \lambda_1)^2}{(\lambda_4 - \lambda_2)(\lambda_2 - \lambda_1)}.
\]

The second of these equations implies that \( \operatorname{coth}(\sqrt{3} \beta k) \) is purely imaginary.

Finally, after some computation using addition formulae for hyperbolic trigonometric functions,

\[
u(t,x) = c_1 r \left( \operatorname{coth}(\sqrt{3} \beta k) + \tanh(\sqrt{3} \beta \xi) \right),
\]  

(132)

\[
u(t,x) = c_2 r \left( \operatorname{coth}(\sqrt{3} \beta k) + \tanh(\sqrt{3} \beta \xi) \right),
\]

where

\[
r = \sqrt{3} \beta \exp \left( \frac{3}{2} \beta \xi_1 - \sqrt{3} \beta (\operatorname{coth}(\sqrt{3} \beta k) \xi_1) \right),
\]

(133)

\[
\xi = x + \frac{3}{2} (\lambda_1 + 2 \lambda_2 + \lambda_4 t + \xi_0).
\]
and $c_1$ and $c_2$ are arbitrary complex constants such that $|c_1|^2 + |c_2|^2 = 1$. This formula is the defocusing–defocusing analog of the focusing–focusing Manakov soliton (121) and, after a straightforward computation, agrees with the dark soliton formula in Eq. (126).

6.1.4. Focusing–defocusing case

The reality conditions in the focusing–defocusing case allow for all three possibilities,

(i) $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$, all real and distinct,
(ii) $\lambda_1 < \lambda_2$, real and distinct and one complex-conjugate pair $\lambda_3 = \lambda_4 = c + id$,
(iii) $\lambda_1 = \lambda_2^* = a + ib$ and $\lambda_3 = \lambda_4^* = c + id$, two complex-conjugate pairs.

However, unlike the focusing–focusing and defocusing–defocusing cases, the graph of the cubic polynomial $y = p(x) = 4x^3 - 2gx - g_3$ is translated upwards as $t_4 > 0$ increases, since $\sigma_1\sigma_3 = -1$.

The soliton limit occurs when the local minimum of $p(x)$ is zero, viz., $p\left(\sqrt{\frac{g_3}{2}}\right) = 0$, so

$$l_4 = \frac{1}{36} \left(\frac{g_2^3}{3^2} + g_0\right).$$

However, because of the condition that $\Delta > 0$, this limit cannot be obtained in all three possibilities listed above without further restrictions. In particular, $\Delta > 0$ if and only if

$$p \left(3 - \frac{1}{4}f_1^2\right) = -9(5^2 - 4l_4) = -9\Delta < 0.$$ 

So the soliton limit can occur if and only if

$$p \left(3 - \frac{1}{4}f_1^2\right) \leq p\left(\sqrt{\frac{g_3}{12}}\right) = p\left(-\frac{g_2}{3}\right) = 0.$$ 

Thus, based on the reality conditions in Section 5.3, in case (i), the soliton limit can occur if and only if

$$3 - \frac{1}{4}f_1^2 \leq -\frac{g_2}{3}. $$

In case (ii), the soliton limit can occur if and only if either

$$3 - \frac{1}{4}f_1^2 \leq -\frac{g_2}{3},$$

or

$$3 - \frac{1}{4}f_1^2 = \sqrt{\frac{g_2}{12}}.$$ 

In case (iii), the soliton limit can occur if and only if

$$3 - \frac{1}{4}f_1^2 \leq \sqrt{\frac{g_2}{12}}.$$ 

In all of these cases the expression for the soliton limit for the signed sum of the squared amplitudes from Eq. (109) is the same,

$$|v|^2 - |u|^2 = \sqrt{\frac{g_2}{12}} - 3\left(\alpha - \frac{1}{4}f_1^2\right) - 3\left(\frac{g_2}{12}\right) \text{sech}^2 \chi(t, x),$$

where

$$\chi(t, x) = \left(\frac{3g_2}{4}\right)^{-1/4} (x - 3f_1t + \xi_0).$$

Thus in case (i) and the first part of case (ii), the soliton limit for $|v|^2 - |u|^2$ is a dark soliton against a positive background, as in the defocusing–defocusing case; while in case (iii) and the second part of case (ii), the soliton limit for $|u|^2 - |v|^2$ is a bright soliton, as in the focusing–focusing case.

6.2. Small-wave-modulation limit

In all cases, focusing–focusing, defocusing–defocusing and focusing–focusing, the small-wave-modulation limit is given by Eq. (110). Therefore, in all cases, the small-wave-perturbation wavenumber, $\kappa_s$, is

$$\kappa_s^2 = 12g_2,$$ 

and the frequency $\Omega_s$ is

$$\Omega_s = 3f_1k_s.$$ 

It is now shown that $\kappa_s = (12g_2)^{1/4}$ and $\Omega_s = 3f_1k_s$, satisfy the linearized dispersion relation for the Manakov system linearized about a plane-wave limit of $u$ and $v$ as $k^2 \to 0$. In order to find the limit of the expressions in Eq. (76) it is necessary to make use of the limiting expressions [9] for the elliptic functions as the imaginary period $\omega' \to i\infty$, in this particular case $e_1 = 2\sqrt{\frac{g_3}{12}}$ and $e_2 = e_3 = -\sqrt{\frac{g_2}{12}}$ so it is convenient to define $\beta = \sqrt{\frac{g_3}{12}}$ and write the limits as

$$\wp(z) = -\beta + 3\beta \csc(3\beta z),$$

$$\zeta(z) = \beta z + \sqrt{3\beta} \cot(\sqrt{3\beta} z),$$

$$\sigma(z) = \frac{1}{\sqrt{3\beta}} \sin(\sqrt{3\beta} e^{z/\beta^2}).$$

Thus, in the small-wave-modulation limit, the quantity $\kappa$ defined in Eq. (50) must satisfy

$$3\beta \csc^2(\sqrt{3\beta} \kappa) = 3\left(\alpha - \frac{1}{4}f_1^2\right) + \sqrt{\frac{g_3}{12}}.$$ 

Therefore,

$$\cot^2(\sqrt{3\beta} \kappa) = \frac{2}{\sqrt{3g_2}} \left(3\left(\alpha - \frac{1}{4}f_1^2\right) - \frac{g_2}{3}\right) < 0,$$

where inequality (147) is true in all cases where the reality conditions are satisfied, since $3(\alpha - \frac{1}{4}f_1^2)$ is always less than the positive $x = \sqrt{\frac{g_3}{12}}$ value for which $p(x)$ equals the value of its local maximum at $x = -\sqrt{\frac{g_2}{12}}$. In particular, for the focusing–focusing or focusing–defocusing reality conditions where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are non-real complex-conjugate pairs,

$$9\left(\alpha - \frac{1}{4}f_1^2\right)^2 - \frac{g_2}{3} = -\frac{27}{16}((\alpha - c)^2 + 4b^2)\times((\alpha - c)^2 + 4b^2) < 0,$$

implying inequality (147). For the focusing–focusing–defocusing reality conditions where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_3, \lambda_4$ are non-real complex-conjugates, inequality (104) is the same as the inequality in (147) and is necessary for an interval of oscillation to exist. In the case of the defocusing–defocusing and focusing–defocusing reality conditions where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, it was shown that $3(\alpha - \frac{1}{4}f_1^2) < e_3 < 0$, so inequality (147) is obviously true. Thus, in all cases,

$$\xi(\kappa) = \beta \kappa + \sqrt{3\beta} \cot(\sqrt{3\beta} \kappa) = \beta \kappa + i\sqrt{3\beta} \gamma,$$

where

$$\gamma = \pm \cot(\sqrt{3\beta} \kappa) \in \mathbb{R}.$$
After some computation, in the small-wave-modulation limit,
\[ u(t, x) = \gamma_1 e^{i\lambda_1 t + \frac{1}{2} \gamma_1 x + i\theta_1} \frac{\sqrt{3\beta}}{\sin(\sqrt{3\beta} \kappa)} \times \left(e^{\sqrt{3\beta} y} - \rho_1 e^{i\theta_1 - i\sqrt{3\beta} \kappa} \right), \]  
\[ v(t, x) = \gamma_2 e^{i\lambda_2 t + \frac{1}{2} \gamma_2 x + i\theta_2} \frac{\sqrt{3\beta}}{\sin(\sqrt{3\beta} \kappa)} \times \left(e^{\sqrt{3\beta} y} - \rho_2 e^{i\theta_2 - i\sqrt{3\beta} \kappa} \right). \]  
(151)

A pure planewave solution of the Manakov system can be obtained from the above linear combinations by a simplifying choice of the integration constant \( \rho_1 = 0 \), in which case \( \rho_2 \to \infty \) but \( \gamma_1, \gamma_4 \) and \( \rho_2 \gamma_2 \) have finite limits. The resulting small-wave-modulation planewaves are

\[ u_0 = \gamma_1 \sqrt{3\beta} \cosec(\sqrt{3\beta} \kappa) \]  
\[ \times \exp \left( \frac{3i}{2} \lambda_1 + i\sqrt{3\beta} y \right) \xi + 9i\alpha t + i\theta_1, \]  
(152)

\[ v_0 = -\rho_2 \gamma_2 \sqrt{3\beta} \cosec(\sqrt{3\beta} \kappa) \times \exp \left( \frac{3i}{2} \lambda_1 - i\sqrt{3\beta} y \right) \xi + 9i\alpha t + i\theta_2. \]

Upon consideration of all the different cases, the signed sum of the squared amplitudes in the small-wave-modulation planewaves is

\[ \sigma_1 |u_0|^2 + \sigma_2 |v_0|^2 = 3\beta \cosec^2(\sqrt{3\beta} \kappa) \]  
\[ = 3 \left( |\alpha| - \frac{1}{4} |\theta| \right)^2 + \frac{9\alpha^2}{12}, \]  
(153)

as expected from Eq. (110). Finally, the planewave wavenumbers for \( u \) and \( v \) are, in all cases,

\[ k_\pm = \frac{3}{2} \kappa \pm \sqrt{3\beta} |\gamma|. \]  
(154)

The linearized dispersion relation for small disturbances about the planewave solution (152) is found by substitution of

\[ u = u_0(1 + u_1), \]  
\[ v = v_0(1 + v_1), \]  
(155)

into the Manakov system and keeping only linear terms in \( u_1 \) and \( v_1 \). This linear system is solvable in terms of Fourier modes

\[ u_1 = n_1 e^{i(\kappa x - \Omega t)} + m_1 e^{-i(\kappa x - \Omega t)}, \]  
\[ v_1 = n_2 e^{i(\kappa x - \Omega t)} + m_2 e^{-i(\kappa x - \Omega t)}, \]  
(156)

if and only if \( k_\kappa \) and \( \Omega_\kappa \) satisfy the linearized dispersion relation,

\[ w^4 + a_2 w^2 + a_1 w + a_0 = 0, \]  
(157)

where

\[ w = \Omega - 3f \kappa_\kappa, \]  
\[ a_2 = 2k_\kappa^2 (-k_\kappa^2 + 2\sigma_1 |u_0|^2 + 2\sigma_2 |v_0|^2 - 12\beta \gamma^2), \]  
\[ a_1 = 16k_\kappa^2 \sqrt{3\beta} \gamma (\sigma_1 |u_0|^2 - \sigma_2 |v_0|^2), \]  
\[ a_0 = k_\kappa^4 (12\beta \gamma^2 - k_\kappa^2 ) (12\beta \gamma^2 + 4\sigma_1 |u_0|^2 + 4\sigma_2 |v_0|^2 - k_\kappa^2), \]  
\[ = k_\kappa^4 (12\beta \gamma^2 - k_\kappa^2 ) (\sqrt{12} g_{12} - k_\kappa^2). \]  
(158)

Now it is evident that, in all cases, the wavenumber \( k_\kappa \) and frequency \( \Omega_\kappa \) satisfying the small-wave-modulation dispersion relation given by (143) and (144) also satisfy the linearized dispersion relation (157), since if \( \Omega_\kappa = \Omega \) and \( k_\kappa = k_\kappa \), then \( w = 0 \) and \( a_0 = 0 \).

6.2.1. Focusing–focusing case

In the particular case of the focusing–focusing equations, the small-wave-modulation limit occurs when \( e_2 = e_3 = -\sqrt{\frac{2}{12}} \), \( e_1 = 2\sqrt{\frac{2}{12}} \) and

\[ I_4 = \frac{1}{36} \left( \frac{1}{3\sqrt{3}} g_{3/2} - g_{30} \right). \]

The reality conditions imply that

\[ k_\kappa^4 = 12g_{12} = 12 \left( \frac{3}{4} (a - c)^2 + b^2 \right) \]  
\[ + \frac{9}{16} d^2 (d^2 + 2(a - c)^2 + 14b^2) \]  
(159)

and the frequency \( \Omega_\kappa \) is

\[ \Omega_\kappa = 3f_1 \kappa_\kappa = -3(a + c) \kappa_\kappa. \]  
(160)

Upon elimination of \( c \) and keeping only the real solutions, the dispersion relation is

\[ \Omega_\kappa = -6a \kappa_\kappa \pm \kappa_\kappa \sqrt{-9(b^2 + d^2) + \sqrt{k_\kappa^4 - 972b^2 d^2}}, \]  
(161)

provided that the stability condition (due to the well-known long-wave instability [5,6]) is satisfied, viz.,

\[ k_\kappa^4 > 81(b^2 + d^2) + 972b^2 d^2. \]  
(162)

Note that the long-wave instability exists even in the case of the scalar equation, since when \( I_4 = 0 \) the small-wave-modulation limit occurs when \( e_2 = e_3 \), viz., \( bd = 0 \), but the right-hand side of the above inequality is non-zero as long as at least one of \( b \) or \( d \) is non-zero. Also note that the two different values of \( \Omega_\kappa \) in Eq. (161) correspond to two different values of \( f_1 \) and, hence, to two different planewaves.

6.2.2. Defocusing–defocusing case

For the defocusing–defocusing case, the parameters \( \lambda_1 \leq \lambda_2 < \lambda_3 < \lambda_4 \) are real and distinct. Upon elimination of \( \lambda_3 \) from the two Eqs. (143) and (144), there are two real roots (corresponding to two different values of \( f_1 \) and hence two different planewaves) of \( \Omega_\kappa \) provided that the stability condition,

\[ k_\kappa^4 > \frac{243}{4} \left( (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1) \right)^{1/2}, \]  
(163)

is satisfied, to avoid the long-wave instability that arises due to the coupling of the equations [5,6]. Notice, however, that unlike the focusing–focusing case, the long-wave instability disappears in the scalar defocusing case since, for \( I_4 = 0, \lambda_2 = \lambda_1 \) when \( e_2 = e_3 \).

6.2.3. Focusing–defocusing case

For the focusing–defocusing cases (i) and (iii) in Section 5.3, the small-wave-modulation limit, viz., \( e_2 \neq e_3 \), can only occur when \( I_4 = 0 \), since \( y = \rho(x) \) is translated vertically upwards for \( I_4 > 0 \). Thus the stability condition of the small-wave-modulation limit dispersion relation in case (i) shows that the long-wave instability disappears in (163) since \( e_2 = e_3 \) implies \( \lambda_2 = \lambda_1 \). However the long-wave instability persists in case (iii), since \( e_2 = e_3 \) only implies \( bd = 0 \), so one of \( b \) or \( d \) could be non-zero while the other is zero.

In case (ii), upon elimination of \( c \) from Eqs. (143) and (144), if \( (\lambda_2 - \lambda_1)^2 - 4d^2 > 0 \), then for each \( k_\kappa \) there is at least one real planewave with a real \( \Omega_\kappa \). If \( (\lambda_2 - \lambda_1)^2 - 4d^2 < 0 \), then there is a planewave with a real solution for \( \Omega_\kappa \) provided that \( k_\kappa^4 > (4d^2 - (\lambda_2 - \lambda_1)^2)^2 - 243(\lambda_1 - \lambda_2)^2d^2 \).
7. Conclusion

The explicit formula for single-phase solutions of the Manakov system of coupled nonlinear Schrödinger equations is obtained in terms of the Weierstrass sigma function, together with reality conditions for all such solutions which are bounded and have a real quasiperiod. An equation is obtained defining the loci of the auxiliary spectral variables for the bounded elliptic solutions of a real quasiperiod. The auxiliary variables have a straightforward interpretation in terms of the internal polarization of the coupled waveform. The parametrization of the solutions is effective in the sense that all the reality conditions are explicitly defined and the relation of the parameters in the solution to the invariants of the trigonal spectral curve is sufficiently explicit to allow the study of the soliton limit and the small-wave-modulation limit. In particular, it is shown that the Manakov soliton is recovered in the soliton limit and that the small-wave-modulation limit satisfies the linearized dispersion relation for planewave solutions. It is hoped that the effective parametrization of single-phase elliptic solutions developed in this paper will be of use in constructing the corresponding Whitham modulation equations.

Appendix. Elliptic functions

Due to the large number of different notational conventions for theta functions, a self-contained collection of relevant definitions and relations between the elliptic functions and the theta functions is included as an Appendix. The notational conventions used here mainly follow those of Whittaker and Watson [25]. The Weierstrass \( \wp \) function is defined by the equation

\[
\wp(z) = \frac{1}{2} \phi(z) - \frac{1}{2} \phi(z) - e_1, \quad (A.1)
\]

with half-periods \( 2e_1, 2e_2, 2e_3 \) satisfying \( \phi(0) = e_i \) for \( i = 1, 2, 3 \), where \( e_3 < e_2 < e_1 \). Moreover, \( \omega_1, \omega_2, \omega_3 \in \mathbb{R} \), since

\[
z = \int_{\omega(z)}^{\omega(z) + \varphi} \frac{dz}{\sqrt{4(t-e_1)(t-e_2)(t-e_3)}}, \quad (A.2)
\]

and, using the fact that \( \wp \) is an even function, \( \omega_1 \) and \( \omega_3 \) can be chosen so that \( \omega_1 > 0 \) and \( -i\omega_3 > 0 \). The definitions

\[
\omega_1 = \omega_0, \quad \omega_2 = -\omega_0 - \omega_3, \quad \omega_3 = \omega_0', \quad (A.3)
\]

imply that

\[
\tau = \frac{\omega^\prime}{\omega} \quad (A.4)
\]

satisfies \( \tau > 0 \) and is the period ratio in the canonical Jacobi theta function

\[
\theta_3(z) = \sum_{n=-\infty}^{\infty} e^{in\tau z + 2inz}. \quad (A.5)
\]

The remaining three theta functions are defined in terms of \( \theta_3 \) by

\[
\theta_1(z) = -ie^{iz + \frac{i}{2}\tau} \theta_3(z + \frac{\tau}{2}), \quad (A.6)
\]

\[
\theta_2(z) = -ie^{iz + \frac{i}{2}\tau} \theta_3(z + \frac{\tau}{2}), \quad (A.7)
\]

Two important identities that are used in Section 6.1.1 to calculate the soliton limit are

\[
\begin{align*}
\theta_1'(0) &= \theta_2(0)\theta_3(0)\theta_4(0), \\
\sqrt{\epsilon_1 - \epsilon_3} &= \frac{\pi}{2\epsilon_1} \theta_1'(0).
\end{align*}
\]

The Weierstrass functions \( \sigma \) and \( \zeta \) are defined by

\[
\begin{align*}
\varphi(z) &= -\zeta'(z), \\
\zeta(z) &= \frac{\sigma'(z)}{\sigma(z)}. \quad (A.9)
\end{align*}
\]

Also the constants

\[
\eta = \zeta(\omega_0), \quad \eta' = \zeta(\omega_0'), \quad (A.10)
\]

satisfy the relation

\[
\eta\eta' - \eta'\omega = \frac{1}{2}i\pi. \quad (A.11)
\]

The Weierstrass \( \sigma \) function is an odd function with quasi-periods \( 2\omega_0 \) and \( 2\omega_0' \).

\[
\begin{align*}
\sigma(z + 2\omega_0) &= -e^{2iz(\omega_0 + \omega)} \sigma(z), \\
\sigma(z + 2\omega_0') &= -e^{2iz(\omega_0' + \omega)} \sigma(z), \quad (A.12)
\end{align*}
\]

and can be expressed in terms of a Jacobi theta function,

\[
\sigma(z) = \frac{2\omega_0}{\pi} \exp \left( \frac{\eta z^2}{2\omega_0} \right) \theta_1 \left( \frac{\pi Z}{2\omega_0} \right). \quad (A.13)
\]

Also the Jacobi elliptic functions can be expressed in terms of the Jacobi theta functions as

\[
\begin{align*}
\text{sn}(\sqrt{\epsilon_1 - \epsilon_3} z) &= \frac{\theta_4(0)}{\theta_4(0) \theta_4(\frac{\pi Z}{2\omega_0})}, \\
\text{cn}(\sqrt{\epsilon_1 - \epsilon_3} z) &= \frac{\theta_2(0) \theta_4(\frac{\pi Z}{2\omega_0})}{\theta_4(0) \theta_4(\frac{\pi Z}{2\omega_0})}, \quad (A.14)
\end{align*}
\]

\[
\text{dn}(\sqrt{\epsilon_1 - \epsilon_3} z) = \frac{\theta_2(0) \theta_4(\frac{-\pi Z}{2\omega_0})}{\theta_4(0) \theta_4(\frac{-\pi Z}{2\omega_0})},
\]

where the elliptic modulus is given by

\[
k^2 = \frac{\theta_3^2(0)}{\theta_4^2(0)}. \quad (A.15)
\]

References


